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A SOLUTION OF THE ACC BENCHMARK PROBLEM BY COEFFICIENT DIAGRAM METHOD

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Abstract. A controller design method, called Coefficient Diagram Method (CDM), is introduced. By this method the designer can design the characteristic polynomial of the closed loop system efficiently taking a good balance of stability, response, and robustness. By CDM, a solution of the ACC benchmark problem is given, and solutions given by various researchers are compared. Theoretical analysis is made to clarify the robustness trade-off.

Key Words Control system design; control theory; vibration control; control system synthesis; controllers; benchmark problem

係数図法によるACCベンチマーク問題の解析

内容梗概　「係数図法」は制御系の新しい設計法であって、これにより閉ループ系の特性多項式を、安定性・応答性・ロバスト性の協調をとりながら、容易に設計することができる。係数図法によって、American Control Conference で提案されたACCベンチマーク問題の解を求め、この解と今まで多数の研究者によって求められた解との比較と、ロバスト性についての理論的検討を行った。

1. INTRODUCTION

The purpose of this paper is to show the effectiveness of a control design method called "Coefficient Diagram Method (CDM)" by solving the benchmark problem proposed by Wie (1992a) in American Control Conference (ACC).

The CDM is fairly new and not well-known, but its basic philosophy has been known in control community for more than 30 years (Graham, 1953) (Chestnut, 1959) (Kessler, 1960) (Kitamori, 1979) (manabe, 1991) (Tanaka, 1992b). The idea has been successfully used in many fields of industry such as steel mill control (Kessler, 1960), gas turbine control (Tanaka, 1992b), and space craft attitude control (Manabe, 1981, 1994b).

The coefficient diagram is a semi-log diagram where the coefficients of characteristic polynomial are shown in logarithmic scale in the ordinate and the numbers of power corresponding to each coefficient are shown in the abscissa, as shown in Fig. 1. The degree of convexity is a measure of stability. The general inclination of the curve is a measure of response speed. The variation of the shape of the curve is a measure of robustness. Thus the three major characteristics of control system, namely stability, response, and robustness are shown graphically in a single diagram, enabling the designer to make a balanced judgment in the course of his design.

The power of the coefficient diagram method (CDM) lies in that it generates not only non-minimum phase controllers but also unstable controllers when required. Unstable controllers are shown to be very effective in controlling such unstable plants as inverted pendulums with limited number of sensors (Manabe, 1994a). LQG sometimes fails to produce a robust controller for plant with flexibility (poles at the vicinity of the imaginary axis) as pointed by various authors (Edmunds, 1983, and Mills, 1992). CDM produces very robust controllers in such cases. The experience show that only well-designed H∞ controller can be equivalent to CDM controllers.

This paper will first explain the basics of CDM. Then ACC benchmark problem is briefly introduced, and a solution is derived by CDM. Then theoretical analysis is made, and trade-off structure of various requirements is clarified. By so doing, it is found that the CDM solution seems to be the only solution to the problem. Then more detail of solution method of CDM is explained. Finally CDM solution is compared with solutions by various researchers, and it is found to be equivalent to or better than the best solution derived so far (Thompson, 1995). The historical background of CDM is referred to other literature (Manabe, 1994b).

2. BASICS OF CDM

General description of CDM

The salient features of CDM will be summarized as follows;

(1) CDM is is a control system design method, where the coefficient diagram is used as a vehicle to carry the necessary information. In CDM, the characteristic polynomial and the

controller are designed simultaneously with due consideration to the performance specification and constraint imposed to the controller. By so doing, both the merit of open-loop synthetic approach in the classical control, where the practical constraints on the controller is always retained, and the merit of the closed-loop analytic approach in modern control, where performance specification is given at first, is retained. This becomes possible, because, in CDM, design is carried out directly to the coefficients of the characteristic polynomial, which are expressed in closed form of the controller and plant parameters.

(2) The CDM is an algebraic design method over polynomial ring. Instead of transfer function, its denominator and numerator are separately expressed as polynomial of s, for the plant and the controller. By so doing, the ambiguity inherent to the transfer function is avoided, and the same rigor as in the state space representation is maintained. At the same time, the compactness of expression as in the transfer function expression is retained.

(3) The theoretical background of CDM is the sufficient condition of stability by Lipatov (1978). The tradition of Kessler (1960) standard form is inherited and improved.

Definitions and mathematical relations

Because polynomial is used extensively in CDM, some shorthand expression of polynomial is necessary. When characteristic polynomial P(s) is given as

$$P(s) = a_n s^n + \dots + a_1 s + a_0 = \prod_{i=0}^{n} a_i s^i \qquad (1)$$

a coefficient row vector in descending order, a_i, is defined as

$$a_i = [a_n \ \dots \ a_1 \ \ a_0] \qquad (2)$$

The polynomial is expressed in shorthand expression as

$$P(s) = [a_i]\, s^i \qquad (3)$$

The addition of the two polynomials corresponds simply to the addition of the two coefficient vectors. The multiplication of two polynomials corresponds to the convolution of two coefficient vectors. Thus if

$$P3(s) = P1(s) + P2(s)\ , \qquad P4(s) = P1(s)\, P2(s) \qquad (4)$$

then

$$a3_i = a1_i + a2_i\,, \qquad a4_i = conv\,(a1_i\,, a2_i) \qquad (5)$$

In the development of CDM, the mathematical notation used in MATLAB is extensively used. The function "conv" is one example.

In CDM, stability index γ_i, equivalent time constant τ, and stability limit γ_i^* play very important role. They are defined from the coefficients of the characteristic polynomials as follows;

$$\gamma_i = a_i^2 / (a_{i+1}\, a_{i-1}) \qquad i = 1 \sim n-1 \qquad (6)$$

$$\tau = a_1 / a_0 \qquad (7)$$

$$\gamma_i^* = 1/\gamma_{i+1} + 1/\gamma_{i-1}\,, \quad i = 1 \sim n-1, \ \ \gamma_n = \gamma_0 = \infty \qquad (8)$$

By use of these parameters, the coefficient vector of the characteristic polynomial is expressed as follows;

$$a_i = [(a_0 \tau^n / \Gamma\Gamma_{n-1}) \ \ \dots \ \ a_0\tau \ \ a_0] \qquad (9a)$$

where

$$\Gamma_{i-1} = \gamma_{i-1} \ \cdots \ \gamma_2 \gamma_1 \qquad (9b)$$

$$\Gamma\Gamma_{i-1} = \Gamma_{i-1} \ \dots \ \Gamma_2 \Gamma_1 = \gamma_{i-1} \gamma_{i-2}^2 \ \cdots \ \gamma_2^{i-2} \gamma_1^{i-1} \qquad (9c)$$

In other word, the coefficient a_i is given as

$$a_i = a_0 \tau^i / (\gamma_{i-1} \gamma_{i-2}^2 \ \cdots \ \gamma_2^{i-2} \gamma_1^{i-1}) \qquad (9d)$$

Also stability index of higher order is defined as

$$\gamma_{ij} = \frac{a_i^2}{a_{i+j}\, a_{i-j}} = [\prod_{k=1}^{j-1} (\gamma_{i+j-k}\, \gamma_{i-j+k})^k]\, \gamma_i^j \qquad (10a)$$

Especially the stability index of the 2nd order is very useful in later development.

$$\gamma_{i2} = a_i^2 / (a_{i+2}\, a_{i-2}) = \gamma_{i+1} \gamma_i^2 \gamma_{i-1} \qquad (10b)$$

From these equations, a few useful relations among the coefficients and the parameters are obtained as follows;

$$a_{i+1} / a_i = \tau / \Gamma_i = \tau / (\gamma_i \ \dots \ \gamma_2 \gamma_1) \qquad (11a)$$

$$(a_{i-1}\, a_{j+1}) / (a_i\, a_j) = \gamma_{i-1} \gamma_{i-2} \ \cdots \ \gamma_{j+2} \gamma_{j+1}$$

(11b)

Coefficient diagram

When a characteristic polynomial P(s) is given as

$$P(s) = 0.25\, s^5 + s^4 + 2\, s^3 + 2\, s^2 + s + 0.2 \qquad (12a)$$

or the coefficient vector a_i of the characteristic polynomial is expressed as

$$a_i = [0.25 \ \ 1 \ \ 2 \ \ 2 \ \ 1 \ \ 0.2] \qquad (12b)$$

the coefficient diagram is shown in Fig. 1, where coefficient a_i, stability index γ_i, equivalent time constant τ, and stability limit γ_i^* are shown in one figure. The stability index can be graphically obtained as in Fig. 2a. When the curvature of a_i curve becomes large, the system become more stable corresponding to larger γ_i's as shown in Fig. 2b. When the curve a_i is left-end-down, the equivalent time constant τ is small and the response is fast as shown in Fig. 2c.

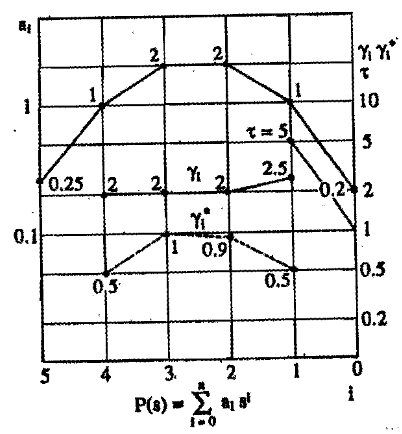
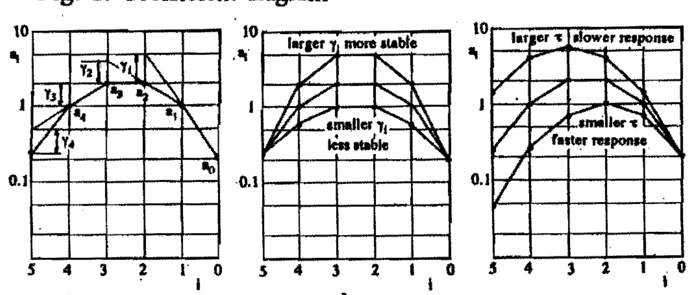


Fig. 1. Coefficient diagram



a b c

Fig. 2. Variation of coefficent diagram

Stability condition used in CDM

The stability conditions are summarized as follows;
By mathematical manipulation of Routh-Hurwitz criterion, the stability condition for the 3rd and 4th order system becomes

$$\gamma_i > \gamma_i^*, \qquad i = 1 \sim n-1 \tag{13}$$

For the system higher than or including 5th degree, the sufficient condition for stability and instability is obtained by Lipatov (1978). The sufficient condition for stability is that either of the following two equations holds for any i.

$$\gamma_i > 1.12\,\gamma_i^*, \quad i = 1 \sim n-1 \tag{14a}$$

$$(\gamma_{i+1}\gamma_i)^{0.5} > 1.47, \ i = 1 \sim n-2 \tag{14b}$$

The sufficient condition for instability is that the following equation holds for some i.

$$(\gamma_{i+1}\gamma_i)^{0.5} < 1, \qquad i = 1 \sim n-2 \tag{15}$$

Eq. (14b) has been quoted in literature (Bose, 1988) (Tanaka, 1992b), but Eq. (14a) has not been quoted in literature to the ahthor's knowledge. In CDM, Eq. (14a) is used, because of its similarity in form to the Routh-Hurwitz criterion for the 3rd and 4th order, and its closeness to necessary and sufficient condition.

Sometimes it is more convenient to express the stability condition in terms of the coefficients of characteristic polynomial. The results are as follows;
Stability condition for the 3rd order

$$a_2 a_1 > a_3 a_0 \tag{16}$$

Stability condition for the 4th order

$$a_2^2 / (a_4 a_0) = \gamma_{2\,2} = \gamma_3 \gamma_2^2 \gamma_1 > 4\,[0.5 + 0.25\,(\mu + 1/\mu)]$$
$$\mu = (a_3/a_1)^2 / (a_4/a_0) = \gamma_3/\gamma_1 \tag{17}$$

Sufficient condition for stability for 5th order and above

$$a_i^2 / (a_{i+2} a_{i-2}) = \gamma_{i\,2} = \gamma_{i+1}\gamma_i^2\gamma_{i-1}$$
$$> 5\,[0.5 + 0.25\,(\mu + 1/\mu)], \quad any\ i = 2 \sim n-2 \tag{18}$$
$$\mu = (a_{i+1}/a_{i-1})^2 / (a_{i+2}/a_{i-2}) = \gamma_{i+1}/\gamma_{i-1}$$

Sufficient condition for instability

$$a_{i+1} a_i < a_{i+2} a_{i-1}, \qquad some\ i = 1 \sim n-2 \tag{19}$$

Stability condition by Lipatov

Lipatov (1978) derived sufficient condition for stability and sufficient condition for instability. Because these conditions constitute the theoretical basis for CDM, brief explanation of the theorems will be made, where notations are modified to harmonize with the expressions of CDM.

The sufficient condition for stability is derived as follows;

(1) First it is proved that the characteristic polynomial of n-th order is stable, if all the partial 5th order polynomials are stable. If coefficient vector expression is used, this can be stated as follows;
Consider a n-th order characteristic polynomial, whose coefficient is a_i

$$a_i = [a_n \ \dots \ a_1 \ \ a_0] \tag{20}$$

If all (n - 6) 5th order polynomials, whose coefficients are aj_i,

$$aj_i = [a_{j+5}\ a_{j+4}\ a_{j+3}\ a_{j+2}\ a_{j+1}\ a_j], \ j = 0 \sim n-5 \tag{21}$$

are stable, the n-th order characteristic polynomial is stable.

(2) The stability condition for the 5-th order polynomial is given by Routh-Hurwitz stability condition. However the formula is fairly complicated. This can be greatly simplified, if sufficient condition is used instead of necessary and sufficient condition, such as Eqs. (14a) and (14b). Thus Eqs. (14a) and (14b) are the sufficient condition for stability for the n-th order characteristic polynomial.

The sufficient condition for instability is proved as follows;

(1) If the 3rd and 4th order polynomial are stable, Eq. (15) does not hold.

(2) Form stable 5th or 6th order polynomial by multiplying a stable 2nd order polynomial to the 3rd or 4th order stable polynomial. By direct calculation, it is found that Eq. (15) does not hold. Thus for any stable polynomial Eq. (15) does not hold. Thus Eq. (15) is sufficient condition for instability.

Standard form of CDM

In CDM, the recommended standard form is

$$\gamma_1 = 2.5, \quad \gamma_{n-1} = \gamma_{n-2} = \ \dots \ = \gamma_2 = 2 \tag{22}$$

The standard form has the favorable characteristics as folows;

(1) When the order of the numerator polynomial is zero, as in type 1 servo, the system has virtually no overshoot. Only negligible overshoot exists for the second and third order. A proper overshoot (about 40%) is guaranteed when the numerator polynomial is selected to form a type 2 servo.

(2) Among the system with the same equivalent time constant τ, the standard form has the shortest settling time. The settling time is about $2.5 \sim 3\,\tau$.

(3) For the same equivalent time constant, the step responses of the standard form show almost equal wave forms irrespective to the order of the characteristic polynomials.

(4) The characteristic roots of lower order have equal decay characteristics with almost equal negative real parts. They are aligned on a vertical line. The characteristic roots for higher order are located within a sector 50 degrees from the negative real axis, and their damping coefficient ζ is larger than 0.64.

(5) The CDM standard form is very easy to remember.

Selection of stability index

In the actual design, the choice of $\gamma_1 = 2.5$, $\gamma_2 = \gamma_3 = 2$ is strongly recommended, but it is not necessary to make $\gamma_4 \sim \gamma_{n-1}$ equal to 2. The condition can be relaxed as

$$\gamma_i > 1.5\,\gamma_i^* \tag{23}$$

In such case, the roots of the higher order become constant decay and the damping coefficient becomes smaller. The robustness decreases slightly, but it is largely offset by narrower bandwidth, increase of design flexibility, and lower order controller. When variation of some parameter is large, the corresponding γ_i may be increased to guarantee desired robustness. Because the essence of the CDM lies in the proper selection of stability indices γ_i's, some experiences are required in actual design, as is true in any design effort.

3. BENCHMARK PROBLEM

Problem statement

The bench mark problem is the control of the two-mass-spring system shown in Fig. 3, which is a generic model of an uncertain dynamical system with a rigid-body mode and one vibration mode. It is assumed that for the nominal system $m_1 = m_2 = 1$ and $k = 1$ with appropriate units and time is in units of seconds. A control force acts on body 1, and the positron of body 2 is measured, resulting in a noncollocated actuator / sensor control problem.

This system can be represented in state-space form as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -k/m_1 & k/m_1 & 0 & 0 \\ k/m_2 & -k/m_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1/m_1 \\ 0 \end{bmatrix} (u + w_1) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1/m_2 \end{bmatrix} w_2 \quad (24a)$$

$$y = x_2 + v \quad (24b)$$

$$z = x_2 \quad (24c)$$

where x_1 and x_2 are the position of body 1 and body 2, respectively; x_3 and x_4 the velocities of body 1 and body 2, respectively; u the control input acting on body 1; y the sensor output; w1 and w2 the plant disturbances acting on body 1 and body 2, respectively; v the sensor noise; and z the output to be controlled (i.e., the performance variable).

There are 4 problems in the benchmark problem, of which Problem 1 and 2 are considered here in order to clarify the nature of the problem. Because the specifications shown in the original problem have some ambiguity, Thompson (1995) combined Problem 1 and 2, and made the specification more concrete.

The proposed problem (problem1 and 2) is as follows;

(1) For a unit impulse disturbance exerted on body 1 or body 2, the controlled output of the nominal system shall not exceed 0.1 after 15 time units.

(2) For the same disturbances the peak control level of the nominal system shall not exceed 1.

(3) The gain margin shall be 6 dB or greater and the phase margin shall be at least 30 deg.

(4) The closed-loop system shall be stable for $0.5 \leq k \leq 2.0$ and $m_1 = m_2 = 1$.

(5). The closed-loop system shall be stable for simultaneous changes $1 - pm \leq k, m_1, m_2 \leq 1 + pm$, $pm = 0.3$.

(6) There shall be reasonable high-frequency sensor noise rejection, performance robustness, and controller complexity.

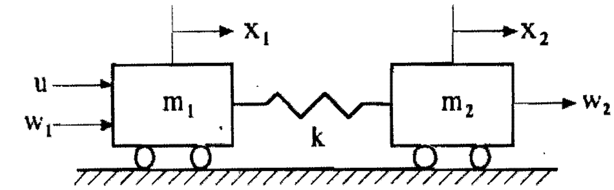


Fig. 3. Two-mass-spring system

Derivation of control system

The plant system matrix will be derived from Eq. (24a) as

$$\begin{bmatrix} m_1 s^2 + k & -k \\ -k & m_2 s^2 + k \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} u + w_1 \\ w_2 \end{bmatrix} \quad (25)$$

This is solved for x_2 as

$$A_p(s) x_2 = u + w_1 + [(m_1/k)s^2 + 1] w_2 \quad (26a)$$

where

$$A_p(s) = (m_1 m_2 / k) s^4 + (m_1 + m_2) s^2 \quad (26b)$$

The standard controller in the CDM is of two degree of freedom type, and expressed as

$$A_c(s) u = B_a(s) y_r - B_c(s)(x_2 + v) \quad (27)$$

where y_r is the reference input for x_2.

The closed-loop system matrix is obtained as

$$\begin{bmatrix} A_p(s) & -1 \\ B_c(s) & A_c(s) \end{bmatrix} \begin{bmatrix} x_2 \\ u \end{bmatrix} = \begin{bmatrix} w_1 + \{(m_1/k)s^2 + 1\} w_2 \\ B_a(s) y_r - B_c(s) v \end{bmatrix} \quad (28)$$

The responses are

$$\begin{bmatrix} x_2 \\ u \end{bmatrix} = \frac{1}{P(s)} \begin{bmatrix} 1 \\ A_p(s) \end{bmatrix} [B_a(s)\, y_r - B_c(s)\, v] + \frac{1}{P(s)} \begin{bmatrix} A_c(s) \\ -B_c(s) \end{bmatrix} [w_1 + \{(m_1/k)\, s^2 + 1\}\, w_2] \quad (29a)$$

where the characteristic polynomial P(s) is given as

$$P(s) = A_c(s) A_p(s) + B_c(s) \quad (29b)$$

Analysis of specification

The specifications given above will be analyzed, and they will be interpreted to the terms of CDM to make the design easier. From problem item (1), the equivalent time constant t = 6 sec is selected, because the settling time is about 2.5 τ and the specified settling time is 15 sec. A controller is called the m / n order controller, if the order of the denominator is n, and that of the numerator m. A 3 / 3 controller is chosen, because this is sufficient to guarantee stability to the 4th order plant, and also is sufficient for the high frequency attenuation. Then $A_c(s)$ and $B_c(s)$ will be assumed as

$$A_c(s) = l_3 s^3 + l_2 s^2 + l_1 s + l_0 \quad (30a)$$

$$B_c(s) = k_3 s^3 + k_2 s^2 + k_1 s + k_0 \quad (30b)$$

where one parameter can be chosen arbitrarily, and l_0 is chosen as

$$l_0 = 1 \quad (30c)$$

and the characteristic polynomial P(s) becomes from Eqs. (26b)(29b)(30a)(30b)

$$P(s) = a_7 s^7 + a_6 s^6 + a_5 s^5 + a_4 s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0$$
$$= [l_3 s^7 + l_2 s^6 + l_1 s^5 + 1 s^4]\alpha + [l_3 s^5 + l_2 s^4 + l_1 s^3 + 1 s^2]\beta + k_3 s^3 + k_2 s^2 + k_1 s + k_0 \quad (31a)$$
$$\alpha = (m_1 m_2 / k), \quad \beta = (m_1 + m_2) \quad (31b)$$

For the nominal case, [α β] is [1 2], and, for the k variation, the limits are [0.5 2] and [2 2]. For the simultaneous variation, the limits are [0.7 1.4], [0.37692 1.4], [1.3 2.6], and [2.4143 2.6]. The gain margin specification of 6 dB can be interpreted as the the variation of [α β] to [0.5 1].

In the controller design, the value of k_2 tends to be minus. Then the decrease of β will produce much decreased value of a_2, which is very detrimental to the stability of the system. For this reason, k_2 must be given special consideration.

From Eq. (29a), it is clear that w_2 requires much more control effort compared with w_1, because of the s^2 term. Considering that w_2 is a unit impulse, the peak value of the u will be about k_3 / a_6 or k_3 / l_2 for the nominal case. Thus this value is taken as a measure of the control effort.

Thus the basis of the CDM design will be summarized as follows;

CDM specification

(1) The equivalent time constant τ is 6 sec.

(2) The controller is the 3 / 3 order.

(3) The nominal value of [α β] is [1 2], and the limits are
[0.5 2], [2 2] for k variation
[0.7 1.4], [0.37692 1.4], [1.3 2.6], [2.4143 2.6] for pm
[0.5 1]. for gain margin of 6 dB.

(4) k_2 is to be made small negative as far as possible.

(5) k_3 / l_2 is to be made to take the value around 1.

In this way the specification is interpreted to the terms of the CDM, and the design will proceed on this basis.

4. DESIGN BY CDM

Preliminary design by CDM

The block diagram of the system is shown in Fig. 4. First design, CDM-1, is made using the standard value for γ_i's and the equivalent time constant τ = 6.

$$\gamma_i = [2 \;\; 2 \;\; 2 \;\; 2 \;\; 2 \;\; 2.5], \quad \tau = 6 \quad (32)$$

The result is shown in Table 1. Some shorthand notations are used as,

$\gamma_i = [\gamma_6 \;\cdots\; \gamma_2 \;\; \gamma_1]$, $\quad a_i = [a_7 \;\cdots\; a_1 \;\; a_0]$

$k_i = [k_3 \;\; k_2 \;\; k_1 \;\; k_0]$, $\quad l_i = [l_3 \;\; l_2 \;\; l_1 \;\; l_0]$

Also the key figure, KF, is defined as

$$KF = [\, k_3 / l_2 \quad k_2] \quad (33a)$$

The first entry of the KF is a measure of the control effort and the second entry ia a measure of robustness. Also in order to summarize the results, the phase margin PM, the gain margin GM, the settling time t_s, the maximum value of the control effort u_{max}, the variation of k, k_{min} / k_{max}, and the simultaneous change limit pm are shown in a row vector FM (Figure of merit) defined as

$$FM = [PM \quad GM \quad t_s \quad u_{max} \quad k_{min} / k_{max} \quad pm] \quad (33b)$$

The bold letters in FM indicate that the specification is not satisfied. Also score for FM, which will be defined later, is shown. This design is robust and satisfy all the specification except the control effort u_{max}, which is fairly large. This design shows u_{max} and robustness is in a trade-off, and one has to be sacrificed for the other.

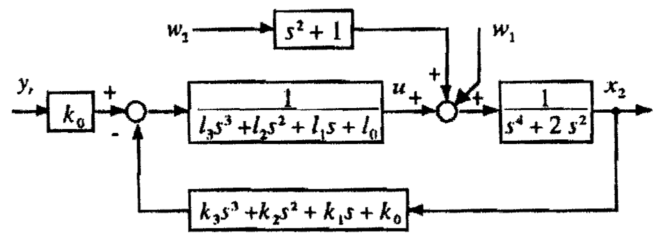


Fig. 4. Block diagram of control system

Improvement of the control effort

In order to decrease u_{max}, it is necessary to make l_2 large and k_3 small. This can be achieved only by reducing the stability index γ_i, especially of the low order. The γ_2 and γ_1 can not be decreased, because then the time response will be deteriorated. Thus γ_5, γ_4, and γ_3 are decreased to 1.5. The lowest value considered is 1.5, because the sufficient condition for stability, Eq.(14a), is always satisfied, if all γ_i's are larger than 1.5. The γ_6 is chosen as 2, because it does not affect l_2.

Thus CDM-2 design is made for

$$\gamma_i = [2 \;\; 1.5 \;\; 1.5 \;\; 1.5 \;\; 2 \;\; 2.5], \quad \tau = 6 \quad (34)$$

The result is shown in Table 1. This design satisfies u_{max} condition, but k_{min} / k_{max} is not satisfied. In order to find the cause for the instability, the stability index γ_i and the stability limit γ_i^* are compared for k variation case.

For k = 0.5,

$$\gamma_i = [2.135 \;\; 1.624 \;\; 2.103 \;\; 0.9255 \;\; 2 \;\; 2.5] \quad (35a)$$
$$\gamma_i^* = [0.6156 \;\; 0.9539 \;\; 1.696 \;\; 0.9755 \;\; 1.481 \;\; 0.5] \quad (35b)$$

Clearly γ_3 is deteriorated. This can only be helped by increasing γ_2, because then γ_3^* is decreased as is clear from Eq. (8). Also for k = 2,

$$\gamma_i = [1.776 \;\; 1.380 \;\; 1.267 \;\; 2.175 \;\; 2 \;\; 2.5] \quad (36a)$$
$$\gamma_i^* = [0.7247 \;\; 1.353 \;\; 1.184 \;\; 1.290 \;\; 0.8598 \;\; 0.5] \quad (36b)$$

Clearly γ_5 is deteriorated. This can be improved by increasing γ_6, because then γ_5^* is decreased.

Improvement of robustness

Now the design CDM-3 is made for

$$\gamma_i = [4 \;\; 1.5 \;\; 1.5 \;\; 1.5 \;\; 2.5 \;\; 2], \quad \tau = 6 \quad (37)$$

The result is shown in Table 1. This design satisfies all the specifications with comfortable margin. The choice of $\gamma_6 = 4$ is more or less arbitrary. However, even if it is increased further, the robustness improvement is not conspicuous. If it is decreased, the robustness decreases but the high frequency attenuation will be better. As will be shown later, k_3, k_2, and l_2 are function of τ_2, where

$$\tau_2 = \tau / (\gamma_2 \gamma_1) \qquad (38)$$

Thus for the same τ, $\gamma_2 = 2.5$ and $\gamma_1 = 2$ give the same τ_2 and thus the same k_3, k_2, and l_2. In the ordinary system, the system without overshoot is preferred and $\gamma_1 = 2.5$ is chosen. In this problem, the overshoot is allowed because the settling time definition is given as in item (1) of the problem. Thus in order to give the maximum value for γ_2, γ_1 is chosen to be 2, which is the smallest value for the good time response wave form.

Final design

Although CDM-3 is a satisfactory design, further improvement can be made by increasing τ, because the settling time has some margin. The final design is made for

$$\gamma_i = [4 \quad 1.5 \quad 1.5 \quad 1.5 \quad 2.5 \quad 2], \quad \tau = 6.4 \qquad (39)$$

The result is shown in Table 1. All the specifications are satisfied with comfortable margin and high score of 8.54 is achieved. The frequency responses of the open-loop transfer function are shown in Fig. 5, and the time responses are shown in Fig. 6.

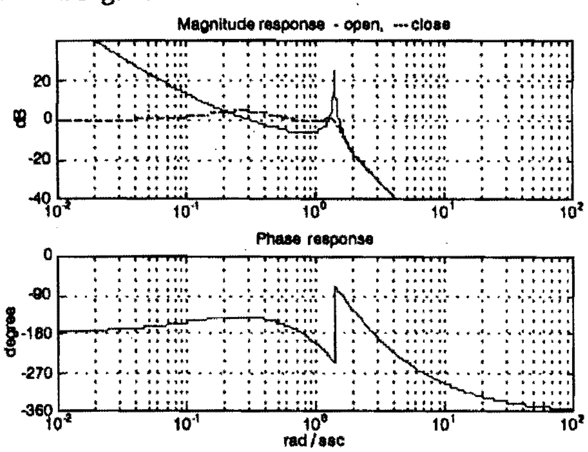


Fig. 5. Frequency response

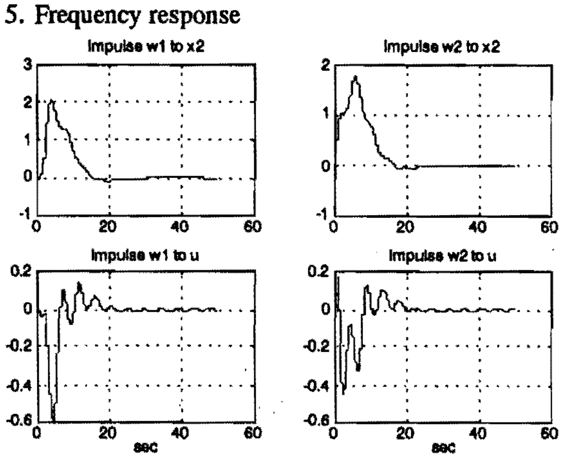


Fig. 6. Time response

5. THEORETICAL ANALYSIS

Theoretical analysis related to control effort and robustness

In ordinary control system design, CDM design is simple and straight forward, because only a few stability indices of low order are chosen in a standard manner and the equivalent time constant becomes only design parameter. However in ACC benchmark problem, the design is complicated, because there is a trade-off between robustness and control effort, and further more this trade-off is very tight due to settling time constraint.

Thus theoretical analysis is required to relate robustness and control effort to CDM design parameters such as stability index γ_i and equivalent time constant τ. Such theoretical analysis is possible due to the close connection of γ_i and τ to the controller and plant parameters, and due to the usage of the sufficient condition for stability by Lipatov (1978). These two features are the most conspicuous points in the CDM, and they are fully utilized in this benchmark problem.

Selection of stability index

First the topic of selecting γ_i from gain margin GM and control effort u_{max} will be discussed. From Eq. (9d), each coefficient of the characteristic polynomial will be shown as

$$a_i = a_2 \tau_2^{i-2} / (\gamma_{i-1} \gamma_{i-2}^2 \cdots \gamma_3^{i-3})$$

$$\tau_2 = \tau / (\gamma_2 \gamma_1), \quad a_2 = a_0 \tau^2 / \gamma_1 \qquad (40)$$

For the nominal case, from Eqs. (31a) and (40), the controller parameters are expressed by γ_i and τ_2 as follows;

$$l_3 = (2 + k_2)\, \tau_2^5 / (\gamma_6 \gamma_5^2 \gamma_4^3 \gamma_3^4) \qquad (41a)$$

$$l_2 = (2 + k_2)\, \tau_2^4 / (\gamma_5 \gamma_4^2 \gamma_3^3) \qquad (41b)$$

$$l_1 + 2 l_3 = (2 + k_2)\, \tau_2^3 / (\gamma_4 \gamma_3^2) \qquad (41c)$$

$$1 + 2 l_2 = (2 + k_2)\, \tau_2^2 / \gamma_3 \qquad (41d)$$

$$k_3 + 2 l_1 = (2 + k_2)\, \tau_2 \qquad (41e)$$

Now if l_2 is eliminated from Eqs. (41b) and (41d), the following relation is derived.

$$1 = (2 + k_2)\, [\tau_2^2 / \gamma_3 - 2 \tau_2^4 / (\gamma_5 \gamma_4^2 \gamma_3^3)] \qquad (42)$$

From this equation, it becomes clear that, for the given τ_2, γ_5, γ_4, and γ_3 are related to k_2. For $\tau = 6$ and $\gamma_2 \gamma_1 = 5$, the τ_2 is equal to 1.2. Thus if $k_2 \geqq -0.3$ is assumed from GM $\geqq$ 6 dB condition as will be explained later and γ_5, γ_4, and γ_3 are chosen to be equal, the solution of Eq. (42) gives

$$\gamma_5 = \gamma_4 = \gamma_3 \leqq 1.4909, \quad \text{or} \quad \geqq 2.3495 \qquad (43)$$

This is the limitation imposed on γ_i by the GM requirement.

Next the $u_{max} \leqq 1$ condition will be considered. In order to satisfy this condition, k_3 / l_2 must take value less than or around 1. If l_1 is eliminated from Eqs. (41c) and (41e), and then divided by Eq. (41b), the following relation is obtained.

$$k_3 / l_2 = [1 - 2 \tau_2^2 / (\gamma_4 \gamma_3^2)] / [\tau_2^3 / (\gamma_5 \gamma_4^2 \gamma_3^3)] + 4\, l_3 / l_2 \qquad (44)$$

By the assumption of $l_3 / l_2 = 0$ (this value is usually very small), $\tau_2 = 1.2$, $\gamma_5 = \gamma_4 = \gamma_3$, and $k_3 / l_2 \leqq 1$, the following relation will be derived.

$$\gamma_5 = \gamma_4 = \gamma_3 \leq 1.5022 \quad (45)$$

From Eqs. (43) and (44), it will be safely concluded that γ_5, γ_4, and γ_3 must take the value around 1.5 in order to satisfy GM and u_{max} condition. Also it should be noted that the condition $\gamma_5 = \gamma_4 = \gamma_3 = 1.5$ satisfies the sufficient condition for stability Eq. (14a). Thus if Eqs. (43) and (45) specify, say, $\gamma_5 = \gamma_4 = \gamma_3 \leq 1.4$, no solution exists for the problem. In other words, this benchmark problem is a very "tight problem", where the solution lies in a very narrow limited region. Eqs. (42) (44), with $l_3 = 0$, and $u_{max} = k_3 / l_2$, are shown in Fig. 7.

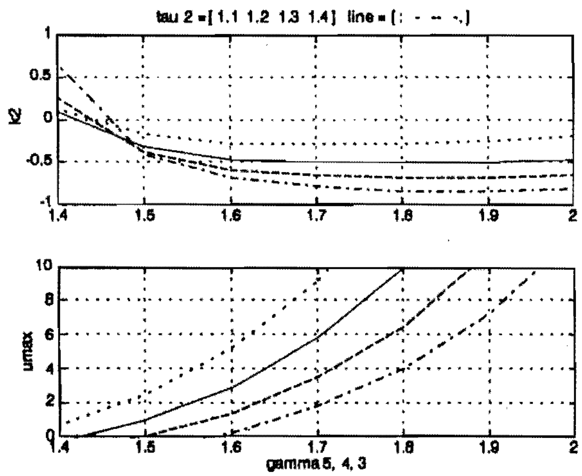


Fig. 7. Relation among design parameters

<u>Trade-off relations</u>

The careful analysis of Fig. 7 helps to clarify the trade-off relations among GM, u_{max}, k_{min} / k_{max} and pm. For the same τ_2, the increase of γ_5, γ_4, and γ_3 necessitates the deterioration of u_{max} and GM. As seen from CDM-1 and CDM-2, the increase of γ_5, γ_4, γ_3 always helps k_{min} / k_{max} robustness due to spring constant k variation. Thus k_{min} / k_{max} is in trade-off relation with GM and u_{max}.

For the same γ_5, γ_4, and γ_3, increase of τ_2 deteriorates GM and improves u_{max}. Thus GM and u_{max} are in trade-off relation. As seen from CDM-3 and CDM-4, increase of τ_2 deteriorates GM, but it improves u_{max} conspicuously and higher score is obtained. The robustness of pm can be interpreted as a combined effect of GM and k_{min} / k_{max}, because the former is related to m_1, m_2 variation, and the latter to k variation. Thus, if higher u_{max} is tolerated, very high robustness can be achieved simply by increasing γ_i and decreasing τ_2.

<u>Relation between GM and k2</u>

Now the relation between GM and k_2 will be examined. From Eqs. (10b) (31a), the second order stability index γ_{22} will be given as follows;

$$\gamma_{22} = a_2^2 / (a_4 a_0) = (2 + k_2)^2 / ((1 + 2 l_2) k_0)$$
$$= \gamma_3 \gamma_2^2 \gamma_1 = 18.75 \quad (46)$$

GM = 6 dB means stability is maintained for the increase of k_2 and k_0 by 2 times. The stability bound is considered around one third of the γ_{22}, 6.25, from Eq. (18). From this observation the following relation is derived.

$$(2 + k_2)^2 / 3 \leq (2 + 2 k_2)^2 / 2 \quad (47a)$$

From this, the bound of k_2 is obtained as follows.

$$k_2 \geq -0.31010 \quad (47b)$$

This is approximate condition for 6 dB GM.

6. DESIGN PROCEDURES

<u>Formal statement of CDM design</u>

The salient feature of CDM is that the characteristic polynomial and the controller are designed simultaneously. By this feature, the trade-off among various requirements is greatly expedited, because some requirements are closely related to the characteristic polynomial, while others are related to the parameter of the controller.

The formal statement of CDM design procedure will be stated as follows;

Given

(1) The plant polynomials, $A_p(s)$, $B_p(s)$

(2) The upper bound of equivalent time constant $\tau < \tau_{max}$

(3) The range of stability index $\gamma_{min} < \gamma < \gamma_{max}$

(4) The order of controller polynomial, n_c, m_c

(5) Some parameters of controller, (usually given by the steady state characteristics) l_i, k_i

Find

(1) Proper equivalent time constant τ

(2) Proper stability index γ_i

(3) The characteristic polynomial $P(s)$

(4) The controller polynomials

denominator $A_c(s)$

feedback numerator $B_c(s)$

reference numerator $B_a(s)$

Considering

(1) Reference tracking characteristics

(2) Disturbance rejection characteristics

(3) Robustness requirement

<u>Design procedures</u>

Thus the design process goes as follows;

(1) Assume the controller in the simplest possible form, such as

$$A_c(s) = l_2 s^2 + l_1 s + l_0 \quad (48a)$$
$$B_c(s) = k_2 s^2 + k_1 s + k_0 \quad (48b)$$
$$B_a(s) = m_1 s + m_0 \quad (48c)$$

Some parameters will be derived from the steady state characteristics.

(2) Solve the following Diophantine equation simultaneously as stated in the formal statement.

$$a_i = a_0 \tau^i / (\gamma_{i-1} \gamma_{i-2}^2 \; \dots \; \gamma_2^{i-2} \gamma_1^{i-1}) \tag{49}$$

$$A_o(s) A_p(s) + B_o(s) B_p(s) = P(s) = a_n s^n + \dots + a_1 s + a_0$$

If solution can not be found, modify the controller assumption and repeat the process.

(3) Design B_a (s) to satisfy the reference tracking characteristics.

It should be noted that the solution of the above Diophantine equation is not straight forward, because known variables and unknown variables are in both left and right side, and also number of equations are not necessarily equal to the number of unknown variables.

Method of solution

Because the solution of the Diophantine equation is not straight forward, following three methods are used together in combination according to the design stage.

(1) Graphical method

At the early design stage, drawing the coefficient diagram by hand is strongly recommended. By so doing, the basic structure of the controller will be designed intuitively.

(2) Use of design form

There is a special design form for CDM. The designer can proceed his design by filling this form.

(3) Use of CAD (Computer Aided Design)

MATLAB M-files are developed to expedite the design. There are three important functions.

"function g2t" gives the most favorable equivalent time constant for the given plant and stability index.

"function g2c" solves the Diophantine equation, and controller parameters are obtained.

"function c2g" calculate various characteristics for given controller and plant. Frequency response, time response, and roots location are shown in figures.

Also there are other functions to relate CDM to LQG. It can be proved that there always exists an augmented state feedback control, which exactly corresponds to a CDM design, if indefinite weight Q is allowed (Hayase, 1973) (ohta, 1991). By use of these functions, the gap between these two approaches is filled effectively.

In ACC benchmark problem, a special M-file, called accfm, is developed, using g2c and c2g, and various characteristics of the control system, including FM and score, are immediately obtained for the given controller.

7. COMPARISON WITH PUBLISHED SOLUTIONS

Scoring system

Thopson introduced a scoring system in his paper (Thompson, 1995, Eq. (21) and Table 2). If the system satisfy the specification, such that

$$\text{FM} = [30 \quad 6 \quad 15 \quad 1 \quad 0.5 / 2 \quad 0.3] \tag{50a}$$

score is 2. If the result exceeds the specification, such that

$$\text{FM} = [40 \quad 10 \quad 9 \quad 0.5 \quad 0.3535 / 2.828 \quad 0.4] \tag{50b}$$

the score becomes 14, the highest value. In this scoring system, a score of zero is good, a score over + 4 is very good, and a score below - 4 indicates the design needs improvement.

Published solutions

The Journal of Guidance, Control, and Dynamics (September-October 1992) presented 11 solutions to the problem. The first 6 solutions use various design techniques such as minimax method (Mills, 1992), game theoretic control (Rhee, 1992), pole placement (Lilja, 1992), quantitative feedback theory (Jayasuriya, 1992), maximum entropy (Collins, 1992), and μ-synthesis (Braatz, 1992). The following 5 solutions use a variety of techniques based on H_∞ theory (Chiang, 1992) (Byrns, 1992) (Wang, 1992) (Adams, 1992) (Wie, 1992).

Thompson (1995) made the specification more concrete and made his own design using classical / H_2 approach, where weights for H_2 solution are obtained from the classical control theory. By so doing, design procedure is more automated, while the rich experiences in classical control are retained.

These 11 solutions and Thompson's solution are tabulated by Thompson (1995). From these solutions, some solutions with high score are shown in Table 2. Chiang's solution is taken from the original paper, which has higher score compared with the one presented by Thompson. Others are the same as those presented by Thompson, with minor correction in numbers.

Comparison of solutions

The results of comparison are as follows;

(1) The CDM design can produce a good controller systematically. It is worthy to notice that even the preliminary design, CDM-1, can produce a fair design. The final design, CDM-4, is better than any previously published design. The controller is 3 / 3 order and simple.

(2) Thmpson's result, Eq. (19), is only controller which satisfies all the specification. This design is similar to CDM-4 in the pattern of γ_i and value of τ. Thus the controllers are similar and FM's are similar, too.

(3) Wie's result is the second highest. This is similar to the minimum-phase controller by Thompson. Both controllers barely missed the specification, and have high score.

(4) Chiang's result comes to the 3rd place in score. It has small GM. The rest of the controllers by Lilja, Braatz, Wang, Byrns, and Adams have about the same score. They have the same tendency of having smaller PM and GM.

(5) Although various methods are presented with various results, the variation of the results is mainly due to the difficulty in interpreting the given specification to the design

parameters such as weights, and the skill to overcome these difficulties. The design methodologies seem to play minor roll in reaching to good design compared with the interpretation of the specification. Thus designer should choose the design methodology which is easiest in the interpretation process.

Table 1 The CDM designs

CDM-1 Standard value design

$\gamma_i = [2\ \ 2\ \ 2\ \ 2\ \ 2\ \ 2.5]$, $\tau = 6$
$k_i = [1.187\ \ -0.4738\ \ 0.6359\ \ 0.1060]$
$l_i = [0.003709\ \ 0.04945\ \ 0.3223\ \ 1]$
KF = [1.187 / 0.04945 - 0.4738] score = - 1.20
FM = [39.5 6.6 12.9 **14.7** 0.35 / 3.25 0.39]

CDM-2 Reduced control effort design

$\gamma_i = [2\ \ 1.5\ \ 1.5\ \ 1.5\ \ 2\ \ 2.5]$, $\tau = 6$
$k_i = [0.5126\ \ -0.3219\ \ 0.6992\ \ 0.1165]$
$l_i = [0.05431\ \ 0.3055\ \ 0.7506\ \ 1]$
KF = [0.5126 / 0.3055 - 0.3219] score = - 0.65
FM = [38.2 6.2 12.8 0.97 **0.53 / 1.74** 0.23]

CDM-3 Robustness recovered design

$\gamma_i = [4\ \ 1.5\ \ 1.5\ \ 1.5\ \ 2.5\ \ 2]$, $\tau = 6$
$k_i = [0.4040\ \ -0.3219\ \ 0.5594\ \ 0.09323]$
$l_i = [0.02716\ \ 0.3055\ \ 0.8049\ \ 1]$
KF = [0.4040 / 0.3055 - 0.3219] score = 6.45
FM = [37.4 7.3 14.1 0.89 0.46 / 2.29 0.35]

CDM-4 Final design

$\gamma_i = [4\ \ 1.5\ \ 1.5\ \ 1.5\ \ 2.5\ \ 2]$, $\tau = 6.4$
$k_i = [0.2039\ \ -0.3895\ \ 0.5033\ \ 0.07864]$
$l_i = [0.03599\ \ 0.3795\ \ 0.9287\ \ 1]$
KF = [0.2039 / 0.3795 - 0.3895] score = 8.54
FM = [35.4 6.2 15.0 0.59 0.47 / 3.45 0.42]

Table 2 Comparison of solutions

Lilja (1992) next eq. after (5) Pole placement

$\gamma_i = [1.029\ \ 1.771\ \ 2.267\ \ 1.999\ \ 2.077]$
$\tau = 8.862$
$k_i = [-0.8964\ \ 0.2586\ \ 0.02919]$
$l_i = [0.6083\ \ 1.178\ \ 1]$
KF = [not applicable] score = 0.26
FM = [**23.8 3.7 29.0** 0.55 0.23 / ∞ 0.35]

Braatz (1992) Eqs. (29) ~ (32) μ-synthesis

$\gamma_i = [2.061\ \ 1.551\ \ 1.556\ \ 1.512\ \ 1.589\ \ 1.592\ \ 2.650]$
$\tau = 6.182$
$k_i = [0.1923\ \ -0.6580\ \ 0.5752\ \ 0.09305]$
$l_i = [0.01346\ \ 0.1097\ \ 0.4068\ \ 0.8871\ \ 1]$
KF = [0.1923 / 0.4068 - 0.6580] score = - 1.9
FM = [**27.2 2.8** 14.1 0.95 **0.57** / 2.50 **0.28**]

Chiang (1992) Eq. (A2) H∞ and pole shifting

$\gamma_i = [1.637\ \ 1.559\ \ 1.438\ \ 1.453\ \ 1.588\ \ 1.813\ \ 2.404]$
$\tau = 7.682$
$k_i = [-0.2431\ \ -0.3151\ \ 0.5273\ \ 0.06864]$
$l_i = [0.09490\ \ 0.4558\ \ 1.148\ \ 1.606\ \ 1]$
KF = [0.2431 / 1.148 - 0.3151] score = 2.8
FM = [30.2 **3.9 15.1** 0.80 0.46 / 3.70 0.36]

Wang (1992) Eq. (32) Observer-based H∞

$\gamma_i = [2.270\ \ 1.731\ \ 1.565\ \ 1.543\ \ 1.641\ \ 1.796\ \ 2.310]$
$\tau = 5.486$
$k_i = [0.4988\ \ -0.5844\ \ 0.5960\ \ 0.1086]$
$l_i = [0.004316\ \ 0.05078\ \ 0.2546\ \ 0.6869\ \ 1]$
KF = [0.4988 / 0.2546 - 0.5844] score = - 0.44
FM = [30.1 **4.1** 11.0 **1.35** 0.50 / 2.10 **0.29**]

Byrns (1992) Eq. (45) H∞ and LTR

$\gamma_i = [0.9455\ \ 2.899\ \ 1.149\ \ 1.777\ \ 1.466\ \ 1.833\ \ 1.850]$
$\tau = 4.986$
$k_i = [0.1157\ \ -0.5840\ \ 0.5255\ \ 0.1054]$
$l_i = [0.01761\ \ 0.09822\ \ 0.5444\ \ 0.98319\ \ 1]$
KF = [0.1157 / 0.5444 - 0.5840] score = - 0.62
FM = [**23.3 3.0** 14.2 0.88 **0.51** / 3.56 0.30]

Adams (1992) Eq. (43) H∞ and LQG

$\gamma_i = [1.737\ \ 1.588\ \ 1.530\ \ 1.737\ \ 1.675\ \ 2.463\ \ 1.9440]$
$\tau = 7.624$
$k_i = [0.2112\ \ -0.9541\ \ 0.2667\ \ 0.3498]$
$l_i = [0.009061\ \ 0.06986\ \ 0.2920\ \ 0.7274\ \ 1]$
KF = [0.2112 / 0.2920 - 0.9541] score = - 2.50
FM = [**24.8 3.4 28.0 1.25** 0.31 / 2.59 0.32]

Wie (1992) Eq. (40) H∞

$\gamma_i = [0.9728\ \ 1.870\ \ 1.194\ \ 1.550\ \ 1.571\ \ 1.910\ \ 2.015]$
$\tau = 8.248$
$k_i = [-0.1324\ \ 0.3533\ \ 0.6005\ \ 0.07280]$
$l_i = [0.5503\ \ 1.418\ \ 2.653\ \ 2.480\ \ 1]$
KF = [- 0.1324 / 2.653 0.3533] score = 6.43
FM = [34.2 6.1 **15.2** 0.57 0.44 / 3.91 0.459]

Thompson (1995) Eq. (19) Classical / H_2

$\gamma_i = [1.929\ \ 2.627\ \ 1.563\ \ 1.552\ \ 1.451\ \ 2.543\ \ 2.020]$
$\tau = 6.263$
$k_i = [0.3263\ \ -0.4548\ \ 0.4985\ \ 0.07960]$
$l_i = [0.002675\ \ 0.03914\ \ 0.2915\ \ 0.7787\ \ 1]$
KF = [0.3263 / 0.2915 - 0.4548] score = 7.36
FM = [35.3 6.0 14.5 0.76 0.45 / 2.81 0.42]

Thompson (1995) Eq. (27) Improvement of Wie's solution

$\gamma_i = [1.026\ \ 1.805\ \ 1.252\ \ 1.451\ \ 1.715\ \ 2.838\ \ 2.120]$
$\tau = 7.908$
$k_i = [-0.09162\ \ 0.006688\ \ 0.5380\ \ 0.06803]$
$l_i = [0.3249\ \ 0.9232\ \ 1.909\ \ 2.082\ \ 1]$
KF = [- 0.09162 / 1.909 0.006688] score = 7.66
FM = [31.7 6.1 14.9 0.56 0.37 / 2.53 0.42]

Thompson (1995) Eq. (28) Minimum-phase

$\gamma_i = [0.8837\ \ 1.957\ \ 1.169\ \ 1.518\ \ 1.610\ \ 1.796\ \ 2.145]$
$\tau = 8.157$
$k_i = [0.7116\ \ 0.7131\ \ 0.08742]$
$l_i = [0.7881\ \ 1.839\ \ 3.278\ \ 2.872\ \ 1]$
KF = [0 / 3.278 0.7116] score = 6.16
FM = [33.6 6.1 13.6 0.59 **0.52** / ∞ 0.37]

8. CONCLUSIONS

The major results of this paper are as follows;

(1) The outline of the coefficient diagram method (CDM) is briefly explained. The CDM is an algebraic approach using only polynomials, where the coefficient diagram is utilized as a vehicle to collectively express the important features of the system, and an improved version or Kessler's standard form and the stability condition of Lipatov constitute the theoretical basis.

(2) In order to evaluate the effectiveness of CDM, a controller for problem 1 and 2 is designed .The result is better or comparable to the best design of H_2 or H∞ control .

(3) By theoretical analysis, the trade-off among requirements is clarified. It is made clear that solution like CDM-4 or one by Thompson is the only possible solution to satisfy the requirement.

Although CDM is a powerful tool for control system design at this stage, further research is needed to make it effective to Multi-Input-Multi-Output system, and to make use of it for design of adaptive control systems

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