

COEFFICIENT DIAGRAM METHOD

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Abstract: A controller design method, called Coefficient Diagram Method (CDM), is introduced. By this method the simplest controller to satisfy the specification can be designed efficiently. The designer can design the controller and the characteristic polynomial of the closed-loop system simultaneously taking a good balance of stability, response, and robustness. *Copyright©1998 IFAC*

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1. INTRODUCTION

In control system design, sensors, actuators, and controllers are considered as the three major components of the system. However design theory must be viewed as the fourth major component, because it affects the controller complexity greatly. Thus theory must be evaluated for its effectiveness in practical application and not for its mathematical authenticity.

All the control system design for linear time invariant dynamic system boils down to proper selection of the characteristic polynomials (denominator polynomials) and proper selection of numerator polynomials for concerned input-output relations. When these polynomials are properly selected, the design of controller transfer function is straight forward, and requires only simple mathematics. The proper selection of the characteristic polynomial is not difficult, if only stability and response are to be satisfied, but it becomes complicated when robustness issue is present. The coefficient diagram method (CDM) (Manabe, 1991) is an answer to this problem.

The paper is organized as follows. In Section 2, a

simple design example is introduced to give the total picture of CDM. The historical background of CDM and its comparison with other control theories are given. In Section 3, basics of CDM, such as mathematical relations, the coefficient diagram, and stability, are given. In Section 4, the general design methodology of CDM is explained with example.

2. BRIEF DESCRIPTION OF CDM

2.1 Simple Example

In order to give the general picture of CDM, a simple design example is given. Fig. 1 is a generic block diagram for a DC motor position control, where position y and velocity v are sensed. The problem is to find velocity gain k_1 and position gain k_0 .

From Fig. 1, the plant equation is obtained in a differential equation form, where $s = d/dt$.

$$(0.25s^3 + 1.25s^2 + s)y = u \quad (1)$$

The controller equation is derived as follows.

$$u = k_0 y_r - (k_1 s + k_0) y \quad (2)$$

By eliminating u , the closed-loop equation is derived.

$$[0.25s^3 + 1.25s^2 + (k_1 + 1)s + k_0]y = k_0 y_r \quad (3)$$

The term preceding y is the characteristic

polynomial denoted as $P(s)$. Thus

$$P(s) = 0.25 s^3 + 1.25 s^2 + (k_1 + 1) s + k_0 \quad (4)$$

$$= \sum_{i=0}^n a_i s^i$$

For stability, all coefficient a_i must be positive, and the Routh stability condition, $a_2 a_1 > a_3 a_0$, must be satisfied. This leads to the the following condition.

$$k_0 > 0. \quad k_1 > 0.2 k_0 - 1 \quad (5)$$

The stability region is shown in Fig. 2a, but, for the controller design, specific values for k_1 and k_0 must be determined.

As will be explained later, stability index γ_i and equivalent time constant τ are defined as follows.

$$\gamma_i = a_i^2 / (a_{i+1} a_{i-1}), \quad i = 1 \sim n-1 \quad (6a)$$

$$\tau = a_1 / a_0 \quad (6b)$$

From Eqs. (4) (6a) (6b), a_i and τ can be expressed in terms of γ_i as follows.

$$a_1 = 1 + k_1 = a_2^2 / (a_3 \gamma_2) = 6.25 / \gamma_2 \quad (7a)$$

$$a_0 = k_0 = a_1^2 / (a_2 \gamma_1) = 31.25 / (\gamma_2^2 \gamma_1) \quad (7b)$$

$$\tau = 0.2 \gamma_2 \gamma_1 \quad (7c)$$

From many design experiences and analytical works, the best choice of stability index is found to be

$$\gamma_i = [\gamma_2 \ \gamma_1] = [2 \ 2.5] \quad (8)$$

This leads to the following design results.

$$a_i = [a_3 \ a_2 \ a_1 \ a_0] = [1 \ 1.25 \ 3.125 \ 3.125] \quad (9a)$$

$$k_i = [k_1 \ k_0] = [2.125 \ 3.125] \quad (9b)$$

$$\tau = 1 \quad (9c)$$

The settling time t_s is about 2.5τ . For this design t_s is about 2.5 sec. Sometimes it may not be necessary to have such fast response. Then increase γ_2 while keeping $\gamma_1 = 2.5$ until a proper τ is obtained. In this case, the results are

$$a_i = [0.25 \ 1.25 \ 6.25 / \gamma_2 \ 12.5 / \gamma_2^2] \quad (10a)$$

$$k_i = [(6.25 / \gamma_2 - 1) \ 12.5 / \gamma_2^2] \quad (10b)$$

$$\tau = 0.5 \gamma_2 \quad (10c)$$

By eliminating γ_2 from Eq. (10a) and using Eqs (7a) (7b) the relation between k_1 and k_0 is obtained.

$$k_0 = 0.32 (1 + k_1)^2 \quad (11)$$

For $\gamma_2 = 6.25, 3.125, 2$, the results are shown in Fig. 2a. This shows the parameter values of controller to be taken in the stability region.

The coefficient diagram is shown in Fig. 2b. The ordinate is the coefficient a_i of the characteristic polynomial in log scale and the abscissa is the order i in the descending order. The convexity at $i = 2$ is more conspicuous for large γ_2 . Larger γ_i

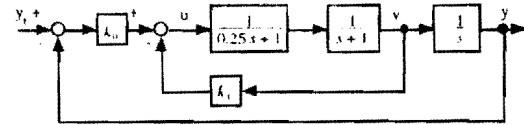


Fig. 1. DC motor position control

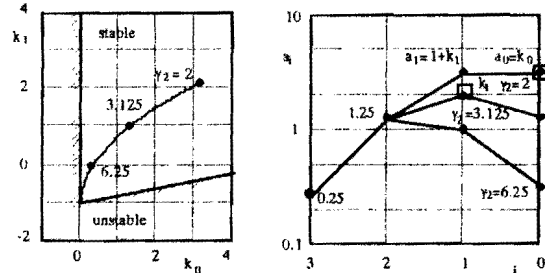


Fig. 2a. Stability region b. Coefficient diagram

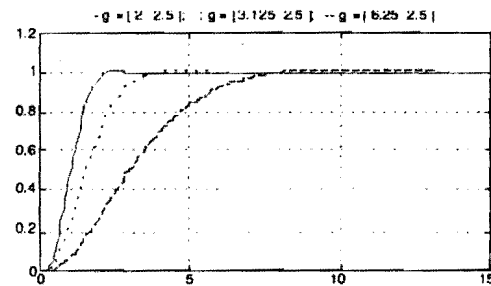


Fig. 3. Step response

corresponds to greater stability as will be later explained. The inclination at the right end corresponds to the equivalent time constant τ , which is the measure of response speed.

The step response for these cases are shown in Fig. 3. The waveforms take similar nice forms, because γ_1 is kept to the optimum value of 2.5, and γ_2 is larger than 2. But the response speeds are different, and the settling time t_s is found to be about 2.5τ .

There are many measure for robustness. One of such measures is the percent variation of a_i to the percent variation of k_i .

$$\Delta a_1 / a_1 = (k_1 / a_1) (\Delta k_1 / k_1) = 0.68 (\Delta k_1 / k_1) \quad (12a)$$

$$\Delta a_0 / a_0 = (k_0 / a_0) (\Delta k_0 / k_0) = 1 (\Delta k_0 / k_0) \quad (12b)$$

For this case, the percent variation of a_i is less than that of k_i . This is the indication of good robustness. This can be easily expressed in the coefficient diagram by plotting k_1 and k_0 with small square symbols besides a_1 and a_0 (shown for $\gamma_2 = 2$).

Thus the coefficient diagram such as Fig. 2b gives the sufficient information about stability, response, and robustness, the three major characteristics of control

systems. The stability is given by curvature, the response is given by inclination, and the robustness is given by the square symbols. From Fig. 2b, it is understood that the increase of γ_2 corresponds to the increase of robustness and stability with the sacrifice of response. The fact that stability, response, and robustness are expressed in the single diagram is the source of the effectiveness of CDM design.

2.2 Historical Background

The CDM has developed over many previous ideas and experiences in control system design. Some of the important topics will be covered in the following.

The first treatment of the polynomial approach is "On governors" by J. C. Maxwell in 1868 and the Routh stability criterion in 1877 (Franklin, 1994), where the stability is analyzed using the coefficients of the characteristic polynomial. However it keeps the original form of stability criterion since then and no further conspicuous effort has been made to make this approach a workable design methodology until Lipatov's work (1978).

In 1950s, the frequency response method was widely used in control system design. During that period, it was commonly recognized that, for good system design, such criteria as the phase margin or gain margin were not sufficient and the frequency characteristics of the open loop transfer function should have proper shape for a fairly wide frequency range (Tustin, 1958).

Chestnut (1951) pointed out in his celebrated book the importance of the relative location of break points and the change of slope at these break points at the straight line approximation of the Bode diagram (gain only) of the open loop transfer function, and he proposed a design method based on these findings. His proposal was very practical, and has been widely used in industry not only in 1950s but even today.

The rule of thumb, such that the straight line approximation of the gain should intersect the 0 dB line at the slope of -20 dB/dec., the change of slope at each break point should be 20 dB/dec., and the break points (time constant) should be separated at least by factor of two, has been widely used in practical design of simple control systems.

For such simple control system, the separation of break points approximately corresponds to the stability index. The rule of thumb that the break

points be separated at least by factor of two roughly corresponds to specifying stability index γ_i to be larger than two. The effort to make this rule applicable to more complex systems have led later to the adoption of stability index rather than the break points, and the adoption of the coefficient diagram rather than Bode diagram, and finally culminated to CDM.

Graham (1953) made intensive research to find the relation between the coefficients of characteristic polynomials and the transient responses, and proposed standard forms for desirable characteristic polynomials. This is commonly called ITAE (Integral Time Absolute Error) standard form. The values of coefficients of this standard form are similar, but a little more oscillatory, compared to the proposed values for CDM.

The shortcoming of ITAE as a design approach is due to its lack of flexibility. Because it gives a standard form for each order of characteristic polynomial, it is very inconvenient when the order varies in the course of design. Because it gives only one standard form, and fails to show the way to modify it when necessary, unreasonably unrobust controller can be designed at certain occasion (Franklin, 1994, p. 534, Ex. 7.21).

Around these time, Kessler (1960) made intensive efforts to establish synthesis (design) procedures for multi-loop control systems, and came out with a standard form, commonly called "Kessler Canonical Multi-loop Structure". The proposed system is more stable compared to ITAE standard form, and, for this reason, has been widely used in the steel mill control. However Kessler's standard form has unnecessary overshoot of 8 %, and it was found that no-overshoot condition can be easily obtained by a small modification of making $\gamma_i = 2.5$ instead of 2. The CDM incorporates this modification.

Various researchers have helped to develop the similar idea in Europe (Brandenburg, 1996) (Zaeh, 1987) (Naslin, 1968). However the term "double ratio" is used instead of stability index γ_i , because it is simply the ratio of ratios of adjacent coefficients.

$$\gamma_i = (a_i / a_{i-1}) / (a_{i-1} / a_i) \quad (13)$$

Kitamori (1979) proposed an improved version of the approach of Graham (1953), where the specification of the characteristic polynomials was given only for the low order, and the flexibility of design was greatly improved.

Stability of control systems can be analyzed by Routh or Hurwitz criterion utilizing coefficients of characteristic polynomials. However in this way the effect of the variation of coefficients on stability is not clearly seen. Lipatov (1978) proposed sufficient conditions for stability and instability. Because of its simplicity, the relation of stability and instability with respect to the coefficients of the characteristic polynomials becomes very clear. These conditions are integrated to the design procedures of CDM.

Especially it becomes a powerful design tool, when it becomes clear that these conditions can be easily shown on the coefficient diagram graphically. It also helps to clarify the meaning of stability index. Thus the designer can graphically design the characteristic polynomial on the coefficient diagram by fully utilizing his intuition and experiences.

In control system design, classical control theory and modern control theory are widely used. But there is another approach called algebraic approach, where polynomials are used instead of transfer functions (Kailath, 1980, p. 306) (Chen, 1987) (Franklin, 1994, p. 564). This method does not say anything about choosing the proper characteristic polynomial for the given problem, and it is usually done by pole assignment. However it greatly simplifies the process of finding controller from the given characteristic polynomial, and this process is adopted in CDM.

As stated above, three features are added to the previous work in deriving the CDM. The first addition is the introduction of the coefficient diagram, where the important features of the control system, namely stability, response and robustness, are represented in a graphical manner, and the understanding of the total system becomes much easier. The second addition is the improvement of the Kessler's standard form, by which the 8% overshoot in Kessler becomes no overshoot. The third addition is the introduction of the Lipatov's sufficient condition for stability in the form compatible with the CDM.

In this way, the CDM has developed from the crude infancy (1991) to a sound control design theory with many successful applications (Manabe, 1994a, 1994b, 1997a, 1998a). Tanaka (1992b) developed independently similar approach by specifying α parameter, which is the reciprocal of stability index, with successful application to a gas turbine design (Tanaka, 1992a). Hori (1994) used the stability index for the design of two-mass resonant system. Such

trend will be further accelerated when the theoretical foundation for the CDM (Manabe, 1997b, 1997c) becomes clearer.

2.3 Comparison with Other Control Theories

The overview of the control design theories are shown in Table 1. One group is the classical control and the other is the modern control. In the middle, there is the third group, called the algebraic approach. These theories are characterized by the mathematical expression used for the system representation, and the way the design proceeds to obtain the controller and the closed-loop transfer function.

As to the mathematical expression, the transfer function (classical control) and the state space (modern control) are commonly used. The transfer function is easy to handle, but it becomes inaccurate when pole-zero cancellation occurs due to uncontrollable or unobservable modes. The state space is accurate and well-suited in machine computation, but manual handling is very difficult.

The third method is the polynomial expression, where the denominator and the numerator of the transfer function is handled as the independent entity. This expression enjoys the easiness of handling of the transfer function together with the rigor of the state space, because it is equivalent to the state space expression in control or observer canonical form.

The control system design problem can be stated as follows; When the plant dynamics and the performance specifications are given, find the controller under some practical limitation together with the closed-loop transfer function such that the performance specifications and controller practical limitation are reasonably satisfied.

Table 1 Overview of control design theories

Classification	Design method	Expression	Approach
Classical control	Frequency response design (Bode / Nyquist)	Transfer function	Open-loop
	Root-locus design		
Algebraic approach	Routh stability criteria	Polynomial	Simultaneous
	Coefficient diagram method	Polynomial	
	Direct method (Pole placement)	Polynomial	
Modern control	Pole placement	State space	Closed-loop
	Optimal control (LQR, LQG)		
	H _∞ control		

One way to proceed in design is to assume the controller under practical limitation first, and then obtain the closed-loop transfer function. After that, it is checked against the performance specification. If it is not satisfactory, modify the controller and repeat the process. This approach is called as "Open-loop approach" and mainly used in the classical control design.

The other way is to find the closed-loop transfer function to meet the performance specification first, and then obtain the controller. If the controller does not satisfy the practical limitation, modify the closed-loop transfer function and repeat the process. This approach is called as "Closed-loop approach" and mainly used in the modern control design.

The third approach is to specify partially the closed-loop transfer function and controller at first, and decide the rest of parameters by design. This approach is called "Simultaneous approach" and used in CDM.

In CDM, the performance specification is rewritten in a few parameters (stability index γ_i and equivalent time constant τ). These parameters specify the closed-loop transfer function. Also these parameters are related to the controller parameters algebraically in explicit form. These features make this approach possible in CDM.

Because of this simultaneous design nature, the designer is able to keep good balance between the rigor of the requirements and the complexity of the controller. Thus the simplest controller to satisfy the specification is realized without much difficulty.

The strength of CDM lies in that, for any plant, minimum phase or non-minimum phase, the simplest and robust controller under practical limitation can be found. Such controller closely agrees with the controllers which are accepted as good controllers in practical application. It is worthy to notice that there always exists a LQR controller with proper weights and state augmentation, which is exactly the same as the CDM controller (Manabe, 1998b).

Simply stated, CDM is an algebraic approach over polynomial ring in the parameter space, where a special diagram called "Coefficient diagram" is used as the vehicle to carry the necessary information, and as the criteria of good design. An improved version of Kessler's standard form and the stability condition of Lipatov constitute the theoretical basis.

3. BASICS OF CDM

3.1 Mathematical Relations

Some mathematical relations extensively used in CDM will be introduced hereafter. The characteristic polynomial is given in the following form.

$$P(s) = a_n s^n + \dots + a_1 s + a_0 = \sum_{i=0}^n a_i s^i \quad (14)$$

The stability index γ_i , the equivalent time constant τ , and stability limit γ_i^* are defined as follows.

$$\gamma_i = a_i^2 / (a_{i+1} a_{i-1}), \quad i = 1 \sim n-1 \quad (15a)$$

$$\tau = a_1 / a_0 \quad (15b)$$

$$\gamma_i^* = 1 / \gamma_{i+1} + 1 / \gamma_{i-1}, \quad \gamma_n = \gamma_0 = \infty \quad (15c)$$

From these equations the following relations are derived.

$$a_{i-1} / a_i = (a_j / a_{j-1}) / (\gamma_i \gamma_{i-1} \dots \gamma_{j+1} \gamma_j) \quad i \geq j \quad (16a)$$

$$a_i = a_0 \tau^i / (\gamma_{i-1} \gamma_{i-2}^2 \dots \gamma_2^{i-2} \gamma_1^{i-1}) \quad (16b)$$

Then characteristic polynomial will be expressed by a_0 , τ , and γ_i as follows.

$$P(s) = a_0 \left[\sum_{i=2}^n \left(\prod_{j=1}^{i-1} 1 / \gamma_{i-j}^j \right) (\tau s)^i \right] + \tau s + 1 \quad (17)$$

The equivalent time constant of the i -th order τ_i and the stability index of the j -th order $\gamma_{i,j}$ are defined as follows.

$$\tau_i = a_{i+1} / a_i = \tau / (\gamma_i \dots \gamma_2 \gamma_1) \quad (18)$$

$$\gamma_{i,j} = a_i^2 / (a_{i+j} a_{i-j}) = \prod_{k=1}^{j-1} (\gamma_{i+j-k} \gamma_{i-j+k})^k \gamma_i^j \quad (19)$$

Thus τ can be considered the equivalent time constant of the 0-th order and γ_i is considered as the stability index of the 1st order. The stability index of the 2nd order is a good measure of stability and is shown below.

$$\gamma_{i,2} = a_i^2 / (a_{i+2} a_{i-2}) = \gamma_{i+1} \gamma_i^2 \gamma_{i-1} \quad (20)$$

3.2 Coefficient Diagram

When a characteristic polynomial is expressed as

$$P(s) = 0.25s^5 + s^4 + 2s^3 + 2s^2 + s + 0.2, \quad (21)$$

then

$$a_i = [0.25 \quad 1 \quad 2 \quad 2 \quad 1 \quad 0.2] \quad (22a)$$

$$\gamma_i = [2 \quad 2 \quad 2 \quad 2.5] \quad (22b)$$

$$\tau = 5 \quad (22c)$$

$$\gamma_i^* = [0.5 \quad 1 \quad 0.9 \quad 0.5] \quad (22d)$$

The coefficient diagram is shown as in Fig. 4, where coefficient a_i is read by the left side scale, and stability index γ_i , equivalent time constant τ , and stability limit γ_i^* are read by the right side scale. The τ is expressed by a line connecting 1 to τ .

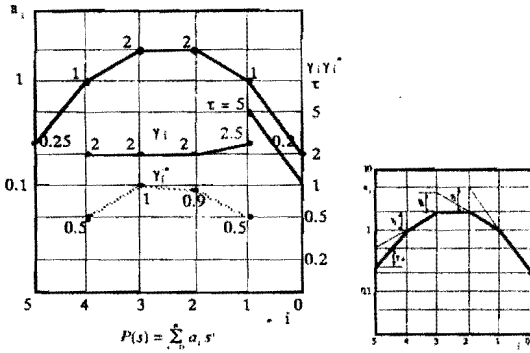


Fig. 4. Coefficient diagram

Fig. 5a. γ_i

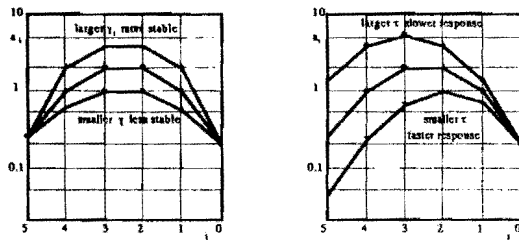


Fig. 5b. Effect of γ_i

c. Effect of τ

The stability index γ_i can be graphically obtained (Fig. 5a). If the curvature of the a_i becomes larger (Fig. 5b), the system becomes more stable, corresponding to larger stability index γ_i . If the a_i curve is left-end down (Fig. 5c), the equivalent time constant τ is small and response is fast. The equivalent time constant τ specifies the response speed.

3.3 Stability Condition

From the Routh-Hurwitz stability criterion, the stability condition for the 3rd order is given as

$$a_2 a_1 > a_3 a_0 \quad (23a)$$

If it is expressed by stability index,

$$\gamma_2 \gamma_1 > 1 \quad (23b)$$

The stability condition for the fourth order is given as

$$a_2 > (a_1 / a_3) a_4 + (a_3 / a_1) a_0 \quad (24a)$$

$$\gamma_2 > \gamma_2^* \quad (24b)$$

For the system higher than or including 5th degree, Lipatov (1978) gave the sufficient condition for stability and instability in several different forms. The conditions most suitable to CDM can be stated as follows;

"The system is stable, if all the partial 4th order polynomials are stable with the margin of 1.12. The system is unstable if some 3rd order polynomial is unstable."

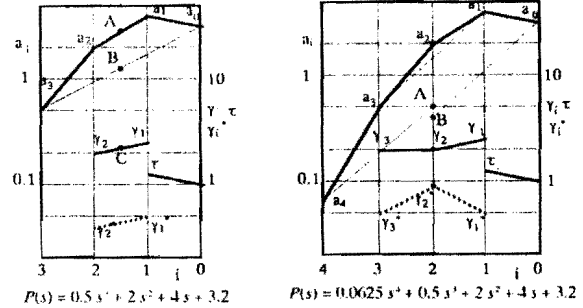


Fig. 6a. 3rd order

b. 4th order

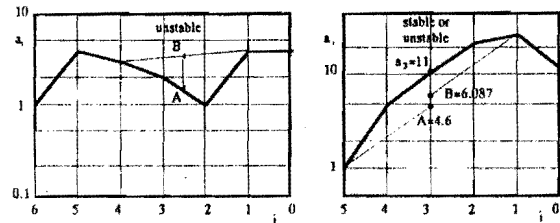


Fig. 7a. 6th order

b. 5th order

Thus the sufficient condition for stability is given as

$$a_i > 1.12 \left[\frac{a_{i-1}}{a_{i+1}} a_{i+2} + \frac{a_{i+1}}{a_{i-1}} a_{i-2} \right] \quad (25a)$$

$$\gamma_i > 1.12 \gamma_i^* \quad \text{for all } i = 2 \sim n - 2 \quad (25b)$$

The sufficient condition for instability is given as

$$a_{i+1} a_i \leq a_{i+2} a_{i-1} \quad (26a)$$

$$\gamma_{i+1} \gamma_i \leq 1, \quad \text{for some } i = 1 \sim n - 2 \quad (26b)$$

These conditions can be graphically expressed in the coefficient diagram. Fig. 6a is a 3rd-order example. Point A is $(a_2 a_1)^{0.5}$ and point B is $(a_3 a_0)^{0.5}$. Thus if A is above B, the system is stable. Point C is $(\gamma_2 \gamma_1)^{0.5}$. If it is above 1, the system is stable.

Fig. 6b is a 4th-order example. Point A is obtained by drawing a line from a_4 in parallel with line $a_3 a_1$. Similarly point B is obtained by drawing a line from a_0 in parallel with line $a_3 a_1$. The stability condition is $a_2 > (A + B)$. The other condition is $\gamma_2 > \gamma_2^*$.

Fig. 7a is a 6th order example (Franklin, 1994, p. 217), where

$$P(s) = s^6 + 4s^5 + 3s^4 + 2s^3 + s^2 + 4s + 4 \quad (27a)$$

By the first glance, the worst points are found to be $[a_4 a_3 a_2 a_1]$, $A < B$, and the system is unstable.

Fig. 7b is for another 5th order example (Franklin, 1994, p. 219), where

$$P(s) = s^5 + 5s^4 + 11s^3 + 23s^2 + 28s + 12 \quad (27b)$$

By the first glance, the worst point is $a_3 = 11$.

because $A = 23/5 = 4.6$, $B = (5/23) 28 = 6.087$, and $A + B = 10.687$, the sufficient condition for stability is

not satisfied. Also by looking at the figure, it is clear that the sufficient condition for instability is not satisfied either. In fact, this system is on the boundary of stability and has imaginary roots at $\pm j2$. It is very interesting to note that $(a_2 / a_4)^{0.5} = 2.145$ is approximately equal to these imaginary roots.

It is clear from Eqs. (15c) (25b) that, if all γ_i s are larger than 1.5, the system is stable. Lipatov (1978) proved, in the process of proving his main theorem, that all roots are real negative, if all γ_i s are larger than 4. From these observation it is safe to say that γ_i should be chosen in a region of 1.5 ~ 4.

3.4 Canonical Open-Loop Transfer Function

For a given characteristic polynomial, the closed-loop transfer function $T(s)$ is given as follows.

$$T(s) = \frac{Q(s)}{P(s)} = \frac{b_m s^m + \dots + b_1 s + b_0}{a_n s^n + \dots + a_1 s + a_0} \quad (28)$$

The canonical open-loop transfer function for $T(s)$ is defined as the open-loop transfer function $G(s)$, whose unity feedback structure produces $T(s)$. Then

$$G(s) = \frac{Q(s)}{[P(s) - Q(s)]} = \frac{\sum_{i=0}^m b_i s^i}{[\sum_{i=m+1}^n a_i s^i + \sum_{i=0}^m (a_i - b_i) s^i]} \quad (29)$$

When $b_i = a_i$ for $i = 0 \sim k$, the system type is said to be $(k + 1)$ (Franklin, 1994, p. 200). Furthermore, if $b_i = 0$ for $i = (k + 1) \sim m$, $G(s)$ is called the canonical open-loop transfer function of system type $(k + 1)$ for characteristic polynomial $P(s)$. Thus the canonical open-loop transfer function of system type 1, $G_1(s)$, for $P(s)$ and its corresponding closed-loop transfer function $T_1(s)$ are given as

$$G_1(s) = a_0 / (a_n s^n + \dots + a_1 s) \quad (30a)$$

$$T_1(s) = a_0 / (a_n s^n + \dots + a_1 s + a_0) \quad (30b)$$

For system type 2,

$$G_2(s) = (a_1 s + a_0) / (a_n s^n + \dots + a_2 s^2) \quad (31a)$$

$$T_2(s) = (a_1 s + a_0) / (a_n s^n + \dots + a_1 s + a_0) \quad (31b)$$

In this way, the canonical open/closed-loop transfer function of a given system-type for a characteristic polynomial $P(s)$ is defined. These canonical transfer functions are helpful to clarify the characteristics of $P(s)$.

Also break point ω_i is defined as

$$\omega_i = a_i / a_{i+1} \quad (32)$$

From Eq. (18), ω_i is found to be the reciprocal of the equivalent time constant of high order τ_i . The ratio of adjacent break points is equal to the stability index γ_i .

$$\gamma_i = \omega_i / \omega_{i-1} \quad (33)$$

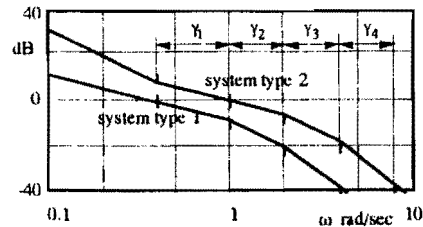


Fig. 8. Canonical open-loop transfer function

Fig. 8 shows an example of Bode diagram for the system type 1 and 2. The straight line approximation (asymptotic representation) of Bode diagram used here is somewhat different from the ordinary way, where the break points are chosen from the poles and zeros of the transfer function, and not the ratio of the coefficients. However this way is more accurate and the relation with the coefficient diagram is closer.

Thus it becomes clear that the coefficient diagram has a one-to-one correspondence with the straight line approximation of Bode diagram of its canonical open-loop transfer function.

3.5 Standard Form

From number of reasons to be explained later, the recommended standard form for CDM is

$$\gamma_{n-1} \sim \gamma_2 = 2, \quad \gamma_1 = 2.5 \quad (34)$$

When $a_0 = 0.4$ and $\tau = 2.5$ are chosen, the characteristic polynomial $P(s)$ is obtained by Eq. (17) in the following simple form.

$$P(s) = 2^{-\frac{(n-2)(n-1)}{2}} s^n + \dots + 2^{-10} s^6 + 2^{-6} s^5 + 2^{-3} s^4 + 0.5 s^3 + s^2 + s + 0.4 \quad (35)$$

The step response of the canonical closed-loop transfer function for the system type 1 and 2 for various orders are given in Fig. 9 and 10. There is virtually no overshoot for the system type 1.

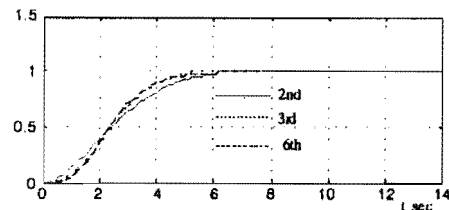


Fig. 9. System type 1

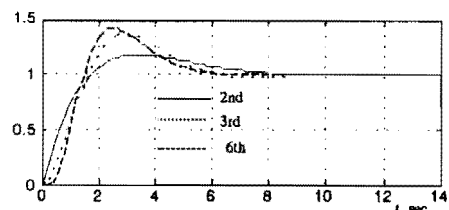


Fig. 10. System type 2

There is an overshoot of about 40% for system type 2. This overshoot is necessary, because the integral of the error for the step response must become zero in system type 2. It is also noticed that the responses are about the same irrespective of the order of the system. Because of this nature, the designer can start from a simple controller and move to more complicated one in addition to the previous design. The settling time is about $2.5 \sim 3\tau$. Many simulation runs show that the standard form has the shortest settling time for the same value of τ .

The pole location is given in Fig. 11. They are listed as follows.

- 2nd order $-0.50000 \pm j 0.38730$
- 3rd order $-0.62273 \pm j 0.82004, -0.75454$
- 4th order $-1.0000 \pm j 1.3764, -1.0000 \pm j 0.32492$
- 5th order $-1.2084 \pm j 0.70569, -1.1377$
 $-2.2228 - j 2.5593$
- 6th order $-1.2867 \pm j 0.74408, -1.1827$
 $-4.4569 \pm j 5.2163, -3.3301$
- 7th order $-1.2843 \pm j 0.73912, -1.1805$
 $-8.9003 \pm j 10.427, -5.8539, -4.5963$
- 8th order $-1.2843 \pm j 0.73925, -1.1806$
 $-17.802 \pm j 20.853, -12.009, -8.3419$
 -4.2969 (36)

It is found that the three lowest order poles are aligned in a vertical line and the two highest order poles are at the point about 49.5 deg from the negative real axis. The rest of the poles are on or close to the negative real axis. For 4th order, all poles are exactly on the vertical line.

It is interesting to note that a 3rd order system with three poles on a vertical line shows a non-decreasing feature for the step response or no overshoot. For example, the transfer function $G(s)$ is given as

$$G(s) = (\beta^2 + 1) / [(s + 1)\{(s + 1)^2 + \beta^2\}] \quad (37a)$$

The first order derivative of its unit step response is the inverse Laplace transform $g(t)$ of $G(s)$, given as

$$g(t) = (1 + 1/\beta^2) e^{-(1 - \cos \beta t)} \quad (37b)$$

Because $g(t)$ is always positive and the unit response is non-decreasing.

A 3rd order system with $\beta^2 = 1.5$ gives a characteristic polynomial $P(s)$ such as

$$P(s) = s^3 + 3s^2 + 4.5s + 2.5 \quad (38a)$$

$$\gamma_i = [2 \quad 2.7] \quad (38b)$$

For this case, the overshoot is zero. If $\gamma_1 = 2.5$ as in the standard form, three poles are not exactly on the vertical line, and the complex poles are a little bit closer to the imaginary axis with the result of a small

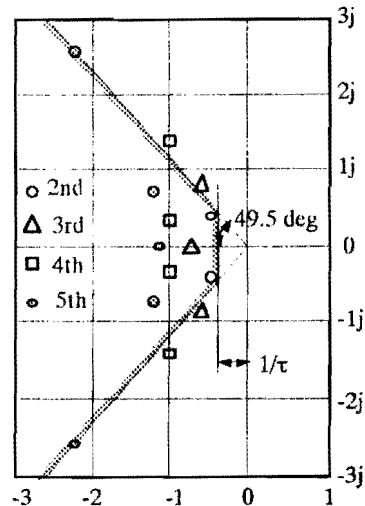


Fig. 11. Pole location

overshoot. The choice of $\gamma_1 = 2.5$ instead of 2.7 is made for the reason of simplicity.

In summary, the standard form has the favorable characteristics as listed below.

- (1) For system type 1, overshoot is almost zero. For system type 2, necessary overshoot of about 40% is realized.
- (2) Among the system with the same equivalent time constant τ , the standard form has the shortest settling time. The settling time is about $2.5 \sim 3\tau$.
- (3) The step responses show almost equal wave forms irrespective to the order of the characteristic polynomials.
- (4) The lower order poles are aligned in a vertical line. The higher order poles are located within a sector 49.5 degrees from the negative real axis, and their damping coefficient ζ is larger than 0.65.
- (5) The CDM standard form is very easy to remember.

In other words, the standard form seems to possess all the characteristics of "good designs" found from experience, such as no overshoot, short settling time, and pole alignment on a vertical line.

Table 2 Comparison of stability index

standard forms	stability index				standard forms	stability index			
	γ_4	γ_3	γ_2	γ_1		γ_4	γ_3	γ_2	γ_1
Binomial			4		ITAE			2	
		3	3				1.424	2.641	
	2.667	2.25	2.667				1.297	2.039	2.144
	2.5	2	2.5		1.568	1.624	1.779	2.102	
Butterworth			2		Kessler			2	
		2	2				2	2	
	2	1.707	2				2	2	2
	2	1.618	1.618	2		2	2	2	
Bessel			3		CDM			2.5	
		2.4	2.5				2	2.5	
		2.222	1.929	2.333			2	2	2.5
	2.143	1.75	1.778	2.25			2	2	2.5

For comparison, stability indices γ_i 's for various standard forms used in the control theory are given in Table 2. It is found that CDM standard is similar to Bessel at the low order, and become similar to binomial at the high order.

3.6 Robustness Consideration

Robustness and stability are completely different concepts. Simply stated, stability concerns where the poles are located, and robustness concerns how fast the poles move to imaginary axis for the variation of parameters.

Stability is specified by the stability index γ_i of the characteristic polynomial, but robustness is only specified after the open loop structure is specified.

As an example, a 3rd order polynomial is given as

$$P(s) = 0.5s^3 + s^2 + s + 0.4 \quad (39a)$$

If the canonical open-loop transfer function of system type 1, $G_1(s)$, is assumed, it becomes

$$G_1(s) = 0.4 / (0.5s^3 + s^2 + s) \quad (39b)$$

For this case phase margin PM = 66.6 deg. For system type 2, $G_2(s)$ becomes

$$G_2(s) = (s + 0.4) / (0.5s^3 + s^2) \quad (39c)$$

For this case, PM = 41.7 deg and robustness is decreased. If a non-minimum controller is used, the open-loop transfer function $G_a(s)$ may become as

$$G_a(s) = (-9s + 0.4) / (0.5s^3 + s^2 + 10s) \quad (39d)$$

For this case, gain margin GM = 1.087 and robustness is extremely poor.

In other words, for the same characteristic polynomial and thus for the same stability, the system can take different robustness. It can become extremely unrobust in some cases.

Conversely any system, which is extremely robust, can be poor in stability. One example is the case, where the open-loop transfer function $G_b(s)$ is given as

$$G_b(s) = k(s^2 + 0.99) / (s^3 + s) \quad (40a)$$

Its characteristic polynomial $P_b(s)$ is given as

$$P_b(s) = s^3 + ks^2 + s + 0.99k \quad (40b)$$

This system is stable for any positive value of k (GM = ∞), and also PM = 90 deg. However the stability is very poor as is clear from Eq. (40b).

Thus in designing the characteristic polynomial, more consideration is required beyond the choice of γ_i .

The traditional design principle of sticking to the minimum-phase controller, wherever possible, with the lowest possible order and with the narrowest possible bandwidth is actually found to be a strong guarantee of robustness.

In the actual design, the choice of $\gamma_1 = 2.5$, $\gamma_2 = \gamma_3 = 2$ is strongly recommended due to stability and response requirement, but it is not necessary to make $\gamma_4 \sim \gamma_{n-1}$ equal to 2. The condition can be relaxed as

$$\gamma_i > 1.5\gamma_i^* \quad (44)$$

With such freedom, designer have the freedom of designing the controller together with the characteristic polynomial, and he can integrate robustness in the the characteristic polynomial with a small sacrifice of stability and response. Because the essence of the CDM lies in the proper selection of stability indices γ_i 's, some experiences are required in actual design, as is true in any design effort.

4. CDM DESIGN

4.1 Mathematical Model

The standard block diagram of the CDM design for a single-input single-output system is shown in Fig. 12. A similar block diagram for multi-input multi-output system can be obtained, but for reasons of simplicity, it will not be treated here.

The plant equation is given as

$$A_p(s)x = u + d \quad (45a)$$

$$y = B_p(s)x \quad (45b)$$

where u , y , and d are input, output, and disturbance. The symbol x is called the basic state variable. $A_p(s)$ and $B_p(s)$ are the denominator and numerator polynomial of the plant transfer function $G_p(s)$.

It will be easily seen that this expression has a direct correspondence with the control canonical form of the state-space expression, and x corresponds to the state variable of the lowest order. All the other states are expressed as the derivatives of x of high order. This

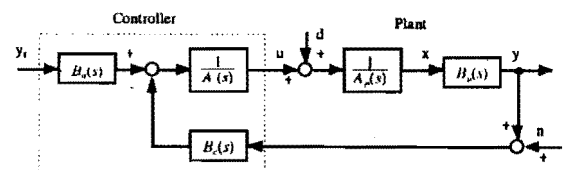


Fig. 12. CDM standard block diagram

form will be called the right polynomial form hereafter, because it corresponds to the right co-prime factorization of the plant transfer function.

Controller equation is given as

$$A_c(s) u = B_a(s) y_r - B_c(s) (y + n), \quad (46)$$

where y_r and n are reference input and noise on the output. $A_c(s)$ is the denominator of the controller transfer function. $B_a(s)$ and $B_c(s)$ are called the reference numerator and the feedback numerator of the controller transfer function. Because the controller transfer function has two numerators, it is called two-degree-of-freedom system.

This expression corresponds to the observer canonical form of the state-space expression. This form will be called the left polynomial form of the controller transfer function.

Elimination of y and u from Eq (46) by Eqs. (45a, b) gives

$$P(s) x = B_a(s) y_r + A_c(s) d - B_c(s) n, \quad (47a)$$

where $P(s)$ is the characteristic polynomial and given as

$$P(s) = A_c(s) A_p(s) + B_c(s) B_p(s). \quad (47b)$$

In a similar manner, equation for y and u are obtained.

$$P(s) y = B_p(s) [B_a(s) y_r + A_c(s) d - B_c(s) n] \quad (47c)$$

$$P(s) u = A_p(s) [B_a(s) y_r - B_c(s) n] - B_c(s) B_p(s) d \quad (47d)$$

Because this system has 3 inputs and 3 outputs, there are 9 transfer functions. But these are related each other. Four basic relations are selected, namely

$$P(s) x = P(0) y_r, \quad (48a)$$

$$P(s) y = B_p(s) B_a(s) y_r, \quad (48b)$$

$$P(s) y = B_p(s) A_c(s) d \quad (48c)$$

$$P(s) (-u) = B_c(s) B_p(s) d. \quad (48d)$$

Eq. (48a) is the response of x to y_r when $B_a(s) = P(0)$, and it corresponds to the canonical closed-loop transfer function of system type 1 for $P(s)$. This equation specifies the characteristic polynomial, and it is a very good measure of stability. Eq. (48b) is for the command following characteristics. Eq. (48c) is for the disturbance rejection characteristics. Eq. (48d) corresponds to the complementary sensitivity function $T(s)$, and it is useful for checking the robustness. In the CDM design, these four basic relations are used for performance specification.

4.2 Analysis of the specification

When the performance specifications are given, they

must be modified to the design specifications. In CDM, the design specifications are as follows.

- (1) The equivalent time constant τ
- (2) The stability indices γ_i for the higher order terms.
The stability indices for the lower order terms are already specified.
- (3) The high frequency attenuation characteristics.
- (4) The low frequency disturbance rejection characteristics.

The items (1) and (2) specifies the characteristic polynomial. The controller is said to be the order m/n , if the order of feedback numerator is m and the order of denominator is n . The item (3) (4) specifies the controller structure, that is, the order m/n and some parameter values.

Usually the rise time, the settling time, the overshoot, and the peak time are used for the time response specification. However from the CDM design point of view, only the settling time t_s is meaningful, because it gives upper bound of τ , where $t_s = 2.5 \sim 3 \tau$. The frequency response specifications are used for the items (3) and (4).

4.3 Design Example

Plant parameters are give as

$$A_p(s) = 0.25 s^3 + 1.25 s^2 + s \quad (49a)$$

$$B_p(s) = 0.1 s + 1. \quad (49b)$$

A 2/2-order controller is to be designed, whose steady state gain is to be 20 due to the disturbance rejection characteristics specification. The controller should have reasonably narrow bandwidth. The command following characteristics of system type 1 is required.

Then the structure of the controller becomes as follows.

$$A_c(s) = l_2 s^2 + l_1 s + 1 \quad (50a)$$

$$B_c(s) = k_2 s^2 + k_1 s + 20 \quad (50b)$$

$$B_a(s) = 20 \quad (50c)$$

In order to make the bandwidth narrow, the highest denominator breakpoint of the controller is limited to twice of that of the plant.

$$l_1 / l_2 = 2 \times 1.25 / 0.25 = 10 \quad (50d)$$

Eqs (50a, b, c, d) have only three degrees of freedom. Thus only $\gamma_1 = 2.5$, $\gamma_2 = \gamma_3 = 2$ can be specified. γ_4 and τ are determined as the result of design.

Now the Diophantine equation is derived as follows.

$$A_c(s) A_p(s) + B_c(s) B_p(s) = P(s) = \sum_{i=0}^n a_i s^i \quad (51a)$$

$$a_i = a_0 \tau^i / (\gamma_{i-1} \gamma_{i-2}^2 \dots \gamma_2^2 \gamma_1^{i-1}) \quad (51b)$$

When Eqs. (49a, b) (50a, b, c, d) are used in Eqs. (51a, b), the following matrix relation is obtained.

$$\begin{bmatrix} 0.375 & 0 & 0 \\ 1.35 & 0.1 & 0 \\ 1 & 1 & 0.1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} l_1 \\ k_3 \\ k_1 \end{bmatrix} = \begin{bmatrix} a_4 \\ a_3 - 0.25 \\ a_2 - 1.25 \\ a_1 - 1 - 2 \end{bmatrix} \quad (52)$$

$$a_4 = a_0 \tau^4 / 12.5, \quad a_3 = a_0 \tau^3 / 12.5, \\ a_1 = a_0 \tau^2 / 2.5, \quad a_0 = 20$$

By multiplying a row vector [-10/3 1 -0.1 0.01] from the left to the Eq. (52), its left hand side vanishes, and an algebraic equation in τ is derived. By solving the equation, $\tau = 2.4248$ is obtained.

Then all the other parameters are determined by Eq. (52). The result is as follows.

$$k_i = [26.488 \quad 45.496 \quad 20] \quad (53a)$$

$$l_i = [1.4750 \quad 14.750 \quad 1] \quad (53b)$$

$$a_i = [0.36876 \quad 5.5313 \quad 22.811 \quad 47.037 \\ 48.496 \quad 20] \quad (53c)$$

$$\gamma_i = [3.6371 \quad 2 \quad 2 \quad 2.5] \quad (53d)$$

$$\tau = 2.4248 \quad (53e)$$

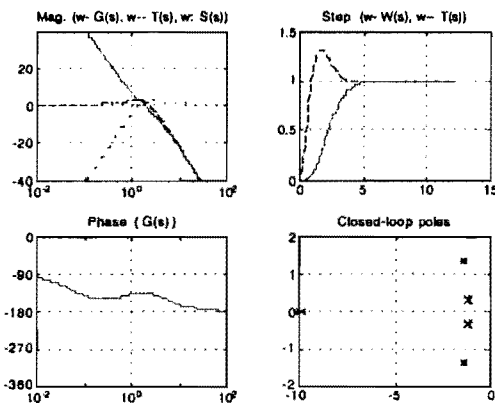
$$\gamma_i^* = [0.5 \quad 0.77494 \quad 0.9 \quad 0.5] \quad (53f)$$

$$s_i = [-9.9385, \quad -1.3679 \pm j \quad 1.3654, \\ -1.1628 \pm j \quad 0.33004] \quad (53g)$$

$$PM = 45.764 \text{ deg (at } 1.7714 \text{ rad/sec)}$$

The vector values such as k_i , l_i , ... , are shown in descending order. s_i 's are the closed-loop poles.

In this design, $\gamma_4 = 3.6371$ is different from the standard form, but the controller becomes simpler. The performance characteristics are shown in Fig. 13.



$$G(s) = B_c(s) B_p(s) / A_c(s) A_p(s), \quad T(s) = B_c(s) B_p(s) / P(s) \\ S(s) = A_c(s) A_p(s) / P(s), \quad W(s) = B_p(s) B_d(s) / P(s)$$

Fig. 13. Performance characteristics

4.4 Summary of Design process

From the design example, the formal statement of the CDM design problem will be summarized as follows;

"Given the plant polynomials, the limitation on the equivalent time constant τ , stability index γ_i , and the controller parameters, find the equivalent time constant τ , the stability index γ_i , the characteristics polynomial $P(s)$, and the controller polynomials, $A_c(s)$, $B_c(s)$ and $B_d(s)$, such that the responses in Eqs. (48b) (48c)(48d) are satisfactory."

The solution process will be as follows;

- (1) Define the plant in the right polynomial form.
- (2) Analyze the performance specifications and derive design specifications for CDM.
- (3) Assume the controller in the simplest possible form. Express it in the left polynomial form.
- (4) Derive the Diophantine equation and solve for unknown variables.
- (5) Make some adjustment to satisfy the performance specification if necessary.

The nature of the problem is to solve the Diophantine equation (51a, b). If a_0 , τ , and γ_i are given beforehand, the problem is exactly the same as the pole allocation design problem. The solution is straight forward, but there is no guarantee of robustness. In the CDM, some of these values are to be determined in the course of solution. The knowns and unknowns are mixed in both sides of the equation. The number of unknowns are not necessarily equal to the number of equations. Eqs. (51a, b) are nonlinear. Because of these reasons, the solution is not straight forward.

There are three methods in the solution. The first method is graphical one. When the coefficient diagram for $A_p(s)$ is drawn, the general structure of the controller and the possible range of the equivalent time constant τ can be graphically obtained, after some experience. The second method is to use the special design form. By filling this form systematically, the solution can be obtained by hand calculation. The third method is the computerization of the second method. Special MATLAB M-files have been developed for this purpose.

In the previous example, graphical method was used to evaluate approximate value of τ . Actual design is done by the second method. The third method is much faster and extensively used in actual design.

5. CONCLUSION

In this paper, the CDM is introduced.

- (1) In CDM, the characteristic polynomial and the controller are designed simultaneously with the help of the coefficient diagram.
- (2) The characteristic polynomial specifies stability and response. The structure of controller guarantees robustness. Thus a simplest controller, which satisfies the stability, response, and robustness requirements, can be designed with ease.

The CDM has a wide application besides designing a satisfactory controller. When combined with LQ design, it gives the analytic method of selecting weights (Manabe, 1998b). The adaptive control is another field of application.

The CDM presented here is for SISO. The method applies to SIMO or MISO, too. The extension to MIMO problem is left for future studies.

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