SUFFICIENT CONDITION FOR STABILITY AND INSTABILITY BY LIPATOV AND ITS APPLICATION TO COEFFICIENT DIAGRAM METHOD

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Abstract. Sufficient condition for stability and instability by Lipatov is the theoretical basis of the Coefficient Diagram Method (CDM), an effective control system design approach. The original paper by Lipatov is expressed and explained in a way compatible with CDM to make its application easier. Although Lipatov's condition is not accurate as that of Routh, it has a great practical value due to its simplicity.

Key Words: Control system design, Control theory, Controllers, Stability, Polynomial.

Lipatpv の安定・不安定十分条件と係数図法への応用

内容梗概 Lipatov の安定・不安定十分条件は、有力な制御系設計法である「係数図法」の理論的根拠になるものである。Lipatov の原論文につき、その係数図法への応用が容易になるように、表現を変更し、また説明を加えた。Lipatpv の条件は Routh の条件のように正確ではないが、形が簡単なので実用性が大変高い。

1. INTRODUCTION

All the control system design for linear time invariant system boils down to proper selection of the characteristic polynomial (denominator polynomial) and proper selection of numerator polynomials for concerned input-output relations. When these polynomials are properly selected, the design of controller transfer function is straight forward, and requires only simple mathematics.

The proper selection of the characteristic polynomial is not difficult, if only stability and response are to be satisfied, but it becomes complicated when robustness issue is present. The coefficient diagram method (CDM) (Manabe, 1998c) is an answer to this problem. Although the effectiveness of CDM is shown in number of examples, it is not widely accepted in control community. One reason for it seems to be the lack of the sound theoretical basis.

The sufficient condition for stability and instability developed by Lipatov (1978), when properly interpreted, gives such theoretical basis. Routh-Hurwitz stability criterion is accurate but requires some computation. It tells whether the system is stable or unstable, but it fails to tell about the degree of stability, which is of practical importance in actual design. Lipatov's condition is only sufficient condition for stability and instability. Thus some ambiguity exists at the border of stability region. But this shortcoming is outweighed by the two merits; namely, simplicity of the expression and capability of indicating the degree of stability.

The idea of CDM is very old (Manabe, 1998c), and various researchers contributed to its development (Chestnut, 1951; Graham, 1953; Tustin, 1958; Kessler, 1960; Naslin, 1968; Kitamori, 1979). The successful application is reported in various fields of control (Zaeh, 1987; Tanaka, 1992a,b; Hori 1994; Brandenburg, 1996). CDM inherited these rich experiences. In addition to it, some improvements are made. The first major improvement is the introduction of the coefficient diagram, whereby graphical design becomes possible like Bode diagram, but with much increased effectiveness. The second improvement is the introduction of the sufficient condition for stability and instability by Lipatov (1978) in truly integrated form. With these improvements, CDM becomes truly practical and theoretically sound control design approach.

This paper is organized as follows. In Section 2, the basics of CDM are explained, and the difference of notations will be clarified. In Section 3, the sufficient conditions for stability and instability by Lipatov are shown in a way compatible with the notation of CDM. In Section 4, the coefficient diagram is introduced and stability condition is graphically presented. In Section 5, the proof of the sufficient condition for instability is made. In section 6, the proof of the sufficient condition for stability is made.

2. BASICS OF CDM

2.1 Mathematical Relations

Some mathematical relations extensively used in CDM will be introduced hereafter. These relations will be freely used in later sections. The characteristic polynomial $P(s)$ is given in the following form.

$$P(s) = a_n s^n + \cdots + a_1 s + a_0 = \sum_{i=0}^{n} a_i s^i \qquad (1)$$

The stability index γ_i, the equivalent time constant τ, and the stability limit γ_i^* are defined as follows.

$$\gamma_i = a_i^2/(a_{i+1}a_{i-1}), \quad i = 1 \sim n-1 \tag{2a}$$

$$\tau = a_1/a_0 \tag{2b}$$

$$\gamma_i^* = 1/\gamma_{i+1} + 1/\gamma_{i-1} \tag{2c}$$

γ_n and γ_0 are defined as ∞.

The equivalent time constant of the i-th order τ_i is defined in the similar manner as τ.

$$\tau_i = a_{i+1}/a_i \tag{3a}$$

By Eq. (2a), the following relation is derived.

$$\tau_i = \tau_{i-1}/\gamma_i = \tau/(\gamma_i \cdots \gamma_2\gamma_1) \tag{3b}$$

By the use of Eqs. (2b)(3a), the coefficient a_i is expressed by τ_i and a_0.

$$a_i = \tau_{i-1}\cdots\tau_1\tau a_0 \tag{4a}$$

By the use of Eq. (3b), this reduces to

$$a_i = a_0\tau^i/(\gamma_{i-1}\gamma_{i-2}^2\cdots\gamma_2^{i-2}\gamma_1^{i-1}), \quad i \geq 2. \tag{4b}$$

Then characteristic polynomial is expressed by a_0, τ, and γ_i as follows.

$$P(s) = a_0[\{\sum_{i=2}^{n}(\prod_{j=1}^{i-1} 1/\gamma_{i-j}^j)(\tau s)^i\} + \tau s + 1] \tag{5}$$

Eqs. (4a) and (4b) can be expressed in a more general form as follows. For $i > j$,

$$a_i = \tau_{i-1}\cdots\tau_{j+1}\tau_j a_j \tag{6a}$$

$$a_i = a_j\tau_j^{i-j}/(\gamma_{i-1}\gamma_{i-2}^2\cdots\gamma_{j+1}^{i-j-1}), \quad i \geq j+2. \tag{6b}$$

For $i < j$,

$$a_i = a_j/(\tau_{j-1}\tau_{j-2}\cdots\tau_i) \tag{6c}$$

$$a_i = a_j\tau_j^{i-j}/(\gamma_j^{j-i}\gamma_{j-1}^{j-i-1}\cdots\gamma_{i+1}), \quad i \leq j-1. \tag{6d}$$

The stability index of the j-th order γ_{ij} is also defined and expressed in terms of γ_i by the use of Eqs. (6b)(6d).

$$\gamma_{ij} = a_i^2/(a_{i+j}a_{i-j}) \tag{7a}$$

$$\gamma_{ij} = \gamma_{i+j-1}\gamma_{i+j-2}^2\cdots\gamma_i^j\cdots\gamma_{i-j+2}^2\gamma_{i-j+1} \tag{7b}$$

The stability index of the 2-nd order is very useful indication of stability.

$$\gamma_{i2} = a_i^2/(a_{i+2}a_{i-2}) = \gamma_{i+1}\gamma_i^2\gamma_{i-1} \tag{7c}$$

The following relation, which is directly derived from Eq. (3b), is also useful.

$$a_i a_j/(a_{i+1}a_{j-1}) = \gamma_i\gamma_{i-1}\cdots\gamma_{j+1}\gamma_j \tag{8}$$

2.2 Differences of Notations

In the original paper by Lipatov (1978), the characteristic polynomial is given as follows.

$$F_n(s) = a_0 s^n + a_1 s^{n-1} + \cdots + a_n \tag{9a}$$

The λ_i parameter is introduced.

$$\lambda_i = (a_{i-1}a_{i+2})/(a_i a_{i+1}) \tag{9b}$$

Considering the difference of notation of characteristic polynomial, λ_i is expressed by stability index γ_i as follows by the use of Eq. (2a).

$$\lambda_{n-i} = 1/(\gamma_i\gamma_{i-1}), \quad i = 2 \sim n-1 \tag{10a}$$

$$\lambda_{n-2} = 1/(\gamma_2\gamma_1) \tag{10b}$$

$$\lambda_1 = 1/(\gamma_{n-1}\gamma_{n-2}) \tag{10c}$$

Because stability is concerned only with λ_i, Lipatov did not introduce any further parameters.

Various authors have already introduced similar parameters used in CDM. Kessler (1960) introduced damping factor α_{n-i} for stability index γ_i, and integration time T_{n-i} for equivalent time constant of high order τ_i. A standard form was suggested where $\tau = 1$, $\gamma_i = 2$ for all i, and $a_0 = 1$. Naslin (1968) introduced characteristic ratio α_i for stability index γ_i, and characteristic pulsatance ω_i, which corresponds to $1/\tau_i$. The recommendation is that γ_i should be around 1.75 and larger than 1.6. Brandenburg (1996) uses double ratio D_{i+1}, which corresponds to $1/\gamma_i$. Tanaka (1992a) uses parameter α_i, which corresponds to $1/\gamma_i$. Kitamori (1979) uses normalized parameter, which corresponds to $a_i/(a_0\tau^i)$.

3. SUFFICIENT CONDITON FOR STABILITY AND INSTABILITY

3.1 Sufficient Condition for Instability

Lipatov gives Theorem 1 and 2 as the sufficient condition for instability.

Theorem 1. If for some i, $1 \leq i \leq n-2$, the condition $\lambda_i > 1$ holds, then a system with characteristic polynomial Eq. (9a) is unstable.

In the form compatible with CDM, the theorem will be stated as follows. The system is unstable if, for some $i = 1 \sim n-2$,

$$\gamma_{i+1}\gamma_i < 1. \tag{11}$$

Because of its simplicity, this condition is used as standard in CDM. The proof is given in Section 5.

Theorem 2. If for some i, $2 \leq i \leq n-2$, the condition $\lambda_{i-1}\lambda_i > C_{ni}$ holds, where

$$C_{ni} = \frac{[n-i+\frac{(-1)^{n+i}-1}{2}][i+\frac{(-1)^i-1}{2}]}{[n-i+\frac{(-1)^{n+i}+3}{2}]^{-1}[i+\frac{(-1)^i+3}{2}]^{-1}}, \tag{12}$$

then a system with characteristic polynomial Eq. (9a) is unstable.

In the form compatible with CDM, the theorem will be stated as follows. The system is unstable if, for some $i = 2 \sim n-2$,

$$\gamma_{i2} < C_{ni}^* \tag{13a}$$

$$C_{ni}^* = \frac{[n-i+1.5+0.5(-1)^{n-i}][i+1.5+0.5(-1)^i]}{[n-i-0.5+0.5(-1)^{n-i}][i-0.5+0.5(-1)^i]}. \tag{13b}$$

This shows that the stability index of 2-nd order γ_{i2} is related to the sufficient condition of instability. This condition is not used in CDM, because of its complexity and limited effectiveness. The proof is in the original paper.

3.2 Sufficient Condition for Stability

Lipatov gives Theorem 3 as the sufficient condition for stability. Two corollaries make the theorem more effective in practical application. The following is the direct quotation from Lipatv's paper.

From the coefficients of the polynomial, we form $n-4$ polynomials of fifth order.

$$F_{5i}(s) = a_i s^5 + a_{i+1} s^4 + \cdots + a_{i+5},$$
$$i = 0 \sim n-5, \quad n \geq 5 \tag{14}$$

Theorem 3. Suppose that the roots of all the polynomials, Eq. (14), are located in the left half-plane. Let

$$\lambda_1 + \lambda_2 < 1, \quad \lambda_{n-3} + \lambda_{n-2} < 1. \tag{15a}$$

Then a system with the characteristic polynomial Eq. (9a) is stable.

Theorem 3 is the main result in this paper and in principle represents a new result in stability theory. The theorem asserts that, for the stability of a linear stationary system of order n, the stability of (n-4) systems of fifth order suffices, and thus enables us to effect a decomposition of a complicated problem into a series of simpler problems whose total complexity is less than that of the original problem, especially for systems of high order.

We remark that in the parameters λ_i the stability conditions for the polynomials Eq. (14) appear very simple:

$$\lambda_{i+1} < 1$$
$$\lambda_{i+2} < (1-\lambda_{i+1})(1-\lambda_{i+3})(1-\lambda_{i+1}\lambda_{i+3})^{-2} \tag{15b}$$
$$i = 0 \sim n-5$$

Using some sufficient conditions for the stability of fifth-order systems, we can simplify even more the sufficient conditions for stability.

Corollary 1. For the coefficients of the polynomial (9a), suppose that the conditions

$$\lambda_i < \lambda^*, \quad i = 1 \sim n-2, \quad n \geq 5, \tag{16a}$$

hold, where λ^* is the real root of the equation

$$\lambda(\lambda+1)^2 = 1. \tag{16b}$$
$$\lambda^* \approx 0.465 \tag{16c}$$

Then a system with the characteristic polynomial Eq. (9a) is stable.

Corollary 2. For the coefficients of the polynomial (9a), suppose that

$$\lambda_i + \lambda_{i+1} < \lambda^{**}, \quad i = 1 \sim n-3, \quad n \geq 5, \tag{17a}$$

where

$$\lambda^{**} = 3/4^{1/3} - 1 \approx 0.89. \tag{17b}$$

Then a system with the characteristic polynomial Eq. (9a) is stable.

The above is the direct quotation of Lipatov's paper. When expressed in the form compatible with CDM, Eqs. (16a, b, c) will be expressed as follows.

$$\sqrt{\gamma_{i+1}\gamma_i} > 1.4546, \quad i = 1 \sim n-2 \tag{18}$$

Eqs. (17a, b) will be as follows.

$$\gamma_i > 1.12374\gamma_i^*, \quad i = 2 \sim n-2, \tag{19}$$

where γ_i^* is the stability limit as defined by Eq. (2c).

In CDM, Corollary 2 or Eq. (19) is used as the sufficient condition for stability, because of its closeness to the necessary condition in practical application. The proof of Theorem 3 in the original paper is modified for improved readability, and is given in Section 6.

4. STABILITY CONDITION EXPRESSED BY COEFFICENT DIAGRAM

4.1 Coefficient Diagram

When the plant/controller transfer functions $G_p(s)$ and $G_c(s)$ are given as

$$G_p(s) = 1/(0.25s^4 + s^3 + 2s^2 + 0.5s) \tag{20a}$$
$$G_c(s) = (k_2 s^2 + k_1 s + k_0)/(l_1 s) \tag{20b}$$
$$k_2 = 1.5, \quad k_1 = 1, \quad k_0 = 0.2, \quad l_1 = 1,$$

the characteristic polynomial $P(s)$ is derived as follows.

$$P(s) = 0.25s^5 + s^4 + 2s^3 + 2s^2 + s + 0.2 \tag{21a}$$

Then the coefficient a_i, stability index γ_i, equivalent time constant τ, and stability limit γ_i^* are expressed in vector form as follows.

$$a_i = [0.25 \quad 1 \quad 2 \quad 2 \quad 1 \quad 0.2] \tag{21b}$$
$$\gamma_i = [2 \quad 2 \quad 2 \quad 2.5] \tag{21c}$$
$$\tau = 5 \tag{21d}$$
$$\gamma_i^* = [0.5 \quad 1 \quad 0.9 \quad 0.5]$$

The coefficient diagram is shown as in Fig. 1, where a_i is read by the left side scale and γ_i, τ, and γ_i^* are read by the right side scale. Equivalent time constant τ is expressed by a line connecting 1 to τ.

From Eq. (3b), stability index γ_i is the ratio of adjacent τ_i s, and it can be obtained graphically as in Fig. 2a. Fig. 2a also shows that, if the curvature of a_i becomes larger, the system becomes more stable, corresponding to larger stability index γ_i. If the a_i curve is left-end down or right-end up (Fig. 2b), equivalent time constant τ is small and response is fast. The equivalent time constant τ specifies the response speed.

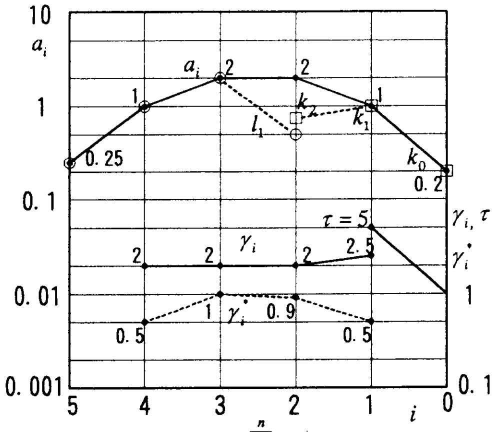
The coefficient diagram is also used for parameter sensitivity analysis and robustness analysis. In this example, the characteristic polynomial $P(s)$ is decomposed into two component polynomials as follows.

$$P(s) = P_{l1}(s) + P_k(s) \tag{22a}$$
$$P_{l1}(s) = l_1(0.25s^5 + s^4 + 2s^3 + 0.5s^2) \tag{22b}$$
$$P_k(s) = k_2 s^2 + k_1 s + k_0 \tag{22c}$$

The auxiliary sensitivity function $T(s)$ is expressed as

$$T(s) = P_k(s)/P(s). \tag{22d}$$

Eq. (22b) is shown on Fig. 1 with small circles and dotted lines. Eq. (22c) is shown with small squares and dotted lines. Designer can visually assess the deformation of the



$P(s) = \sum_{i=0}^{n} a_i s^i$

Fig. 1. Coefficient diagram

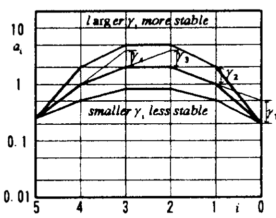


Fig. 2a. Effect of γ_I

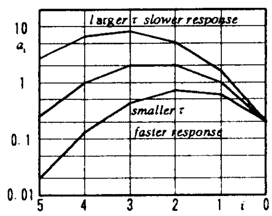
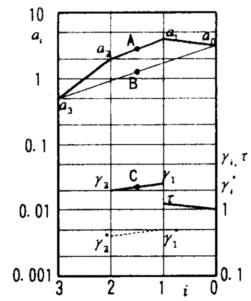
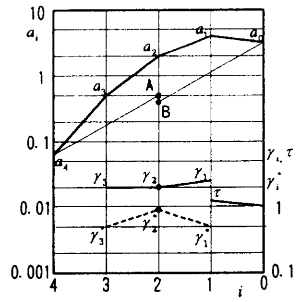


Fig. 2b. Effect of τ



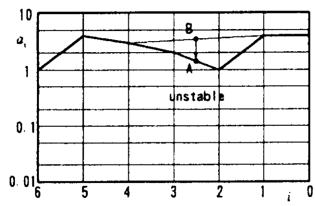
$P(s) = 0.5s^3 + 2s^2 + 4s + 3.2$

Fig. 3a. 3-rd order



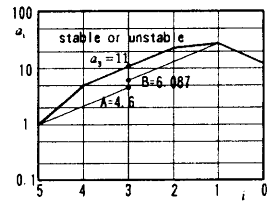
$P(s) = 0.0625s^4 + 0.5s^3 + 2s^2 + 4s + 3.2$

Fig. 3b. 4-th order



$P(s) = s^6 + 4s^5 + 3s^4 + 2s^3 + s^2 + 4s + 4$

Fig. 4a. 6-th order



$P(s) = s^5 + 5s^4 + 11s^3 + 23s^2 + 28s + 12$

Fig. 4b. 5-th order

coefficient diagram due to the parameter change of k_2, k_1, and k_0. Then he can visualize the variation of stability and response. Also from Eq. (22d), it is clear that robustness can be analyzed by comparison of coefficients a_i and k_i at the coefficient diagram.

As explained above, the coefficient diagram indicates stability, response, and robustness (three major properties in control design) in a single diagram, enabling the designer to grasp the total picture of the control system. At present, Bode diagram is used for this purpose. However coefficient diagram is more accurate and easy to use in actual design.

4.2 Stability Condition

From the Routh-Hurwitz stability criterion, the stability condition for the third order is given as

$$a_2 a_1 > a_3 a_0 . \qquad (23a)$$

If it is expressed by stability index,

$$\gamma_2 \gamma_1 > 1 . \qquad (23b)$$

The stability condition for the fourth order is given as

$$a_2 > (a_1 / a_3) a_4 + (a_3 / a_1) a_0 . \qquad (24a)$$

If it is expressed by stability index and stability limit,

$$\gamma_2 > \gamma_2^* . \qquad (24b)$$

For the system higher than or equal to 5-th order, Lipatov's stability conditions Eqs. (11) and (19) will be used. If the partial i-th order polynomial is defined as a i-th order polynomial whose coefficients are taken from i+1 successive coefficients of the original polynomial as Eq. (14), the stability condition is summarized in Theorem 4.

Theorem 4. The system is stable, if all the partial 4-th order polynomials are stable with the margin of 1.12. The system is unstable if some partial 3-rd order polynomial is unstable.

Thus the sufficient condition for stability is given as

$$a_i > 1.12 [\frac{a_{i-1}}{a_{i+1}} a_{i+2} + \frac{a_{i+1}}{a_{i-1}} a_{i-2}] \qquad (25a)$$

$$\gamma_i > 1.12 \gamma_i^* , \quad for\ all\ i = 2 \sim n-2 . \qquad (25b)$$

The sufficient condition for instability is given as

$$a_{i+1} a_i \le a_{i+2} a_{i-1} \qquad (26a)$$

$$\gamma_{i+1} \gamma_i \le 1 , \quad for\ some\ i = 1 \sim n-2 . \qquad (26b)$$

When coefficient diagram is used, the designer can find the worst (least convex) 3-rd order or 4-th order partial polynomial at a first glance, and application of the theorem gives the answer immediately.

These stability conditions can be graphically expressed in the coefficient diagram. Fig. 3a is a 3-rd order example. Point A is $(a_2 a_1)^{0.5}$ and point B is $(a_3 a_0)^{0.5}$. Thus if A is above B, the system is stable. Point C is $(\gamma_2 \gamma_1)^{0.5}$. If it is above 1, the system is stable.

Fig. 3b is a 4-th order example. Point A is obtained by drawing a line from a_4 in parallel with line a_3 a_1. Similarly point B is obtained by drawing a line from a_0 in parallel with line a_3 a_1. The stability condition is $a_2 >$ (A + B). The other condition is $\gamma_2 > \gamma_2^*$.

Fig. 4a is a 6-th order example (Franklin, 1994, p.217), where

$$P(s) = s^6 + 4s^5 + 3s^4 + 2s^3 + s^2 + 4s + 4 . \qquad (27a)$$

By the first glance, the worst points are found to be $[a_4\ a_3\ a_2\ a_1]$ and A < B. Thus the system is unstable.

Fig. 4b is another 5-th order example (Franklin, 1994, p.219), where

$$P(s) = s^5 + 5s^4 + 11s^3 + 23s^2 + 28s + 12 \ . \qquad (27b)$$

By the first glance, the worst point is $a_3 = 11$. Because A = 23/5 = 4.6, B = (5/23)28 = 6.087, and A + B = 10.687, the sufficient condition for stability is not satisfied. Also looking at the figure, it is clear that the sufficient condition for instability is not satisfied either. In fact, this system is on the boundary of stability and has imaginary roots at $\pm j2$. It is very interesting to note that

$$(a_2 / a_4)^{0.5} = 2.145 \qquad (27c)$$

is approximately equal to these imaginary roots.

4.3 Selection of Stability Index

In CDM, the designer designs the controller such that the coefficient diagram has favorable shape, where the selection of stability index is of utmost importance.

Stability condition by Lipatov as expressed in Theorem 4 seems to suggest two important properties. First, in order to indicate the degree of stability, γ_i / γ_i^* ratio may be used as a good measure. Secondly, if all γ_i s are larger than 1.5, the system is stable. Lipatov also expressed third important property in the process of proving Theorem 3 (explained in Section 6). If all γ_i s are larger than 4, all roots are distinct, negative, and real.

From these reason, stability index γ_i is usually chosen in a region 1.5~4. In actual practice, a standard form is suggested in CDM, based on practical experiences, as follows.

$$\gamma_{n-1}, \cdots, \gamma_3, \gamma_2 = 2, \quad \gamma_1 = 2.5 \qquad (28a)$$

Or in a more relaxed form,

$$\gamma_i / \gamma_i^* > 1.5, \quad for\ i = 4 \sim n-1 \qquad (28b)$$

$$\gamma_3 = \gamma_2 = 2, \quad \gamma_1 = 2.5 \ .$$

It is very interesting to note that, for the standard form of Eq. (28a), γ_i / γ_i^* ratio is 2 for most i.

5. PROOF OF SUFFICIENT CONDITION FOR INSTABILITY

The proof of the sufficient condition for instability (Theorem 1) is given in Appendix of the original paper. The proof will be reiterated here using CDM notation.

From a stable n-th order polynomial $P(s)$, form a stable (n+2)-th order polynomial $P^*(s)$ by multiplying a stable second order polynomial as follows.

$$P(s) = a_n s^n + \cdots + a_1 s + a_0 \qquad (29a)$$

$$P^*(s) = (s^2 + \alpha s + \beta) P(s), \quad \alpha, \beta > 0 \qquad (29b)$$

The coefficient of new polynomial is given as follows.

$$P^*(s) = a_{n+2}^* s^{n+2} + \cdots + a_1^* s + a_0^* \qquad (30a)$$

$$a_i^* = a_{i-2} + \alpha a_{i-1} + \beta a_i \ , \quad i = 0 \sim n+2 \ , \qquad (30b)$$

where $a_{n+2}, a_{n+1}, a_{-1}, a_{-2}$ are considered 0. Now calculate the value D defined as follows, where γ_i^* is the stability index of the new polynomial.

$$D = (\gamma_{i+1}^* \gamma_i^* - 1) a_{i+2}^* a_{i-1}^* = a_{i+1}^* a_i^* - a_{i+2}^* a_{i-1}^* \qquad (31a)$$

$$i = 1 \sim n$$

By Eq. (30b), D is calculated as follows.

$$\begin{aligned} D = & (a_{i-1} + \alpha a_i + \beta a_{i+1})(a_{i-2} + \alpha a_{i-1} + \beta a_i) \\ & - (a_i + \alpha a_{i+1} + \beta a_{i+2})(a_{i-3} + \alpha a_{i-2} + \beta a_{i-1}) \end{aligned} \qquad (31b)$$

$$\begin{aligned} D = & (a_{i-1} a_{i-2} - a_i a_{i-3}) + \alpha^2 (a_i a_{i-1} - a_{i+1} a_{i-2}) \\ & + \beta^2 (a_{i+1} a_i - a_{i+2} a_{i-1}) + \alpha (a_{i-1}^2 - a_{i+1} a_{i-3}) \\ & + \beta (a_{i+1} a_{i-2} - a_{i+2} a_{i-3}) + \alpha\beta (a_i^2 - a_{i+2} a_{i-2}) \end{aligned} \qquad (31c)$$

By Eqs. (7c) and (8), this is expressed in terms of stability index γ_i .

$$\begin{aligned} D = & (\gamma_{i-1} \gamma_{i-2} - 1) a_i a_{i-3} + \alpha^2 (\gamma_i \gamma_{i-1} - 1) a_{i+1} a_{i-2} \\ & + \beta^2 (\gamma_{i+1} \gamma_i - 1) a_{i+2} a_{i-1} + \alpha (\gamma_i \gamma_{i-1}^2 \gamma_{i-2} - 1) a_{i+1} a_{i-3} \\ & + \beta (\gamma_{i+1} \gamma_i \gamma_{i-1} \gamma_{i-2} - 1) a_{i+2} a_{i-3} + \alpha\beta (\gamma_{i+1} \gamma_i^2 \gamma_{i-1} - 1) a_{i+2} a_{i-2} \end{aligned}$$

$$i = 1 \sim n \qquad (31d)$$

By Eq. (30b), if $a_i > 0$ ($i = 0 \sim n$), $a_i^* > 0$ ($i = 0 \sim n+2$). From Eqs. (31c, d), D becomes positive for $i = 1 \sim n$, if $a_i > 0$ ($i = 0 \sim n$), $a_{n+2}, a_{n+1}, a_{-1}, a_{-2} = 0$, and $\gamma_{i+1} \gamma_i > 1$ ($i = 1 \sim n-2$). By Eq. (31a), this leads to the condition $\gamma_{i+1}^* \gamma_i^* > 1$ ($i = 1 \sim n$).

The stable 3-rd and 4-th order polynomial has the property that $\gamma_{i+1} \gamma_i > 1$. Thus the stable 5-th and 6-th order polynomial must have the same property $\gamma_{i+1} \gamma_i > 1$, and so on. The lack of it immediately leads to the conclusion that the polynomial is unstable. This completes the proof.

6. PROOF OF SUFFICIENT CONDITION FOR STABILITY

6.1 Stability Theorem

The detailed proof of Theorem 3 is given in Appendix of the original paper. In order to facilitate the understanding of the proof, some basics of the stability theory are reviewed hereafter.

A stable characteristic polynomial $P(s)$ is composed of stable 1-st order and 2-nd order polynomials.

$$\begin{aligned} P(s) & = a_n s^n + \cdots + a_1 s + a_0 \\ & = a_n [\prod_{j=1}^{L} (s + \sigma_j)][\prod_{k=1}^{M} (s^2 + \alpha_k s + \beta_k)] \end{aligned} \qquad (32)$$

$$\sigma_j, \alpha_k, \beta_k > 0, \quad n = L + 2M \ ,$$

where a_n is arbitrarily chosen to be positive. Because the coefficient a_i is the product sum of the positive numbers, it must be always positive. This is the necessary condition for stability, and a_i is considered positive hereafter, because otherwise the system is found to be unstable automatically.

For $s = j\omega$, the argument of P(s) becomes

$$\arg P(j\omega) = \sum_{j=1}^{L} \arg(\sigma_j + j\omega) + \sum_{k=1}^{M} [(\beta_k - \omega^2) + j\alpha_k \omega] \ . \qquad (33)$$

Thus if ω increases from 0 to ∞, $\arg P(j\omega)$ increases monotonically from 0 to $(\pi/2)n$ radians. In unstable system, $\arg P(j\infty)$ is defined as $(\pi/2)n$. Then $\arg P(j0)$ becomes πn_+ radians, where n_+ is the number of unstable roots. Thus for ω increase from 0 to ∞, $\arg P(j\omega)$ changes from πn_+ to $(\pi/2)n$ radians in unstable system. This relation is summarized in Theorem 5, which is an elementary expression of argument principle of Cauchy and Sturm, and called as Theorem of Cremer-Leonhard-Michailov (Mansour, 1994).

Theorem 5. The polynomial $P(s)$ is stable, if and only if $\arg P(j\omega)$ increases monotonically from 0 to $(\pi/2)n$ radians for the ω increase from 0 to ∞.

Now from Theorem 5, the stability condition of Hermite-Bieler will be derived. For $s = j\omega$, $P(s)$ will be decomposed into real and imaginary part, where $x = -\omega^2$.

$$P(j\omega) = R(x) + j\omega I(x) \tag{34}$$

The sum of the orders of $R(x)$ and $I(x)$ is $n-1$. From Theorem 5, the following observation is made for stable polynomial. $R(0)$ and $I(0)$ must be positive. For decrease of x from 0 to $-\infty$, $R(x)$ becomes 0 first, then $I(x)$ becomes 0, and this process repeats until the total number of zero crossing becomes $n-1$. This condition is summarized in the stability condition by Hermite-Bieler as Theorem 6 (Mansour, 1994).

Theorem 6. The polynomial $P(s)$ is stable, if and only if $R(0)$ and $I(0)$ are positive, and $R(x)$ and $I(x)$ have simple real negative alternating roots. The root closest to zero is of $R(x)$.

Theorem 6 will be used extensively in the proof of the sufficient condition of stability explained hereafter.

6.2 Proof of the Sufficient Condition for Stability

The proof of the original paper is some what modified to facilitate its understanding in conformity with CDM. The proof proceeds in 4 steps. In Step 1 (Condition for negative real roots), it is shown that, for a polynomial of positive real coefficient, the sufficient condition that all roots are distinct, negative, and real is

$$\gamma_i > 4, \quad i = 1 \sim n-1. \tag{35}$$

The proof is given in Appendix A1.

In Step 2 (Interval for negative real roots), it is shown that such negative roots s_i lies in some interval determined by a_i and γ_i of $P(s)$.

$$-\mu_i < s_i < -\eta_i, \quad i = 1 \sim n \tag{36}$$
$$\mu_i = 2(a_{i-1}/a_i)[1+\sqrt{1-4/\gamma_i}]^{-1}$$
$$\eta_i = 0.5(a_{i-1}/a_i)[1+\sqrt{1-4/\gamma_{i-1}}]$$
$$\gamma_n = \gamma_0 = \infty$$

The proof is given in Appendix A2.

In Step 3 (Stability condition of the partial 5-th order polynomial), the following relations are derived from the stability condition of the partial 5-th order polynomial.

$$\gamma_{i2} > 4, \quad i = 2 \sim n-2 \tag{37}$$

$$-0.5(a_i/a_{i+2})\ [1+\sqrt{1-4/\gamma_{i2}}]$$
$$< -2(a_{i-1}/a_{i+1})\ [1+\sqrt{1-4/\gamma_{i+1\,2}}]^{-1} \tag{38}$$
$$i = 2 \sim n-3$$

The derivation is given in Appendix A3.

Step 4 is the main body of proof. The proof is made by the direct application of stability condition of Hermite-Bieler (Theorem 6). For polynomial $P(s)$ of Eq. (32), $R(x)$ and $I(x)$ of Eq. (34) will be given as follows.

$$R(x) = a_m x^{m/2} + \cdots + a_2 x + a_0 \tag{39a}$$
$$m = n\,(n\,even),\ n-1\,(n\,odd)$$
$$I(x) = a_{l+1} x^{l/2} + \cdots + a_3 x + a_1 \tag{39b}$$
$$l = n-2\,(n\,even),\ n-1\,(n\,odd)$$

For brevity, only n even case will be considered. It is clear that $R(0)$ and $I(0)$ are positive. Because $\gamma_{i2} > 4$ for $i = 2 \sim n-2$, all roots of $R(x)$ and $I(X)$ are found to be distinct, negative, and real from Step 1. It should be noted that the stability index of $R(x)$ and $I(x)$ corresponds to the stability index of the 2-nd order of $P(s)$. All roots of $R(x)$ and $I(X)$, x_{Ri} and x_{Ii}, are listed as follows. The left end is negative largest.

$$x_{Ri} = x_{Rn/2}, \cdots, x_{R2}, x_{R1} \tag{40a}$$
$$x_{Ii} = x_{I\,n/2-1}, \cdots, x_{I2}, x_{I1} \tag{40b}$$

By proper interpretation of Eq. (36) in Step 2, the intervals for the roots $x_{R1}, x_{I1}, x_{R2}, x_{I2}, x_{I\,n/2-1}$ and $x_{R\,n/2}$ are given as follows.

$$-2(a_0/a_2)[1+\sqrt{1-4/\gamma_{22}}]^{-1} < x_{R1} < -(a_0/a_2) \tag{41a}$$
$$-2(a_1/a_3)[1+\sqrt{1-4/\gamma_{32}}]^{-1} < x_{I1} < -(a_1/a_3) \tag{41b}$$
$$-2(a_2/a_4)[1+\sqrt{1-4/\gamma_{42}}]^{-1}$$
$$< x_{R2} < -0.5(a_2/a_4)[1+\sqrt{1-4/\gamma_{22}}] \tag{41c}$$
$$-2(a_3/a_5)[1+\sqrt{1-4/\gamma_{52}}]^{-1}$$
$$< x_{I2} < -0.5(a_3/a_5)[1+\sqrt{1-4/\gamma_{32}}] \tag{41d}$$
$$-(a_{n-3}/a_{n-1}) < x_{I\,n/2-1}$$
$$< -0.5(a_{n-3}/a_{n-1})[1+\sqrt{1-4/\gamma_{n-3\,2}}] \tag{41e}$$
$$-(a_{n-2}/a_n) < x_{Rn/2}$$
$$< -0.5(a_{n-2}/a_n)[1+\sqrt{1-4/\gamma_{n-2\,2}}] \tag{41f}$$

First it is found that $x_{I1} < x_{R1}$, because

$$(a_1/a_3) > 2(a_0/a_2)[1+\sqrt{1-4/\gamma_{22}}]^{-1}. \tag{42a}$$

Eq. (42a) is verified in the following manner. It is modified by replacing a_i and γ_{22} with γ_i.

$$0.5(\gamma_2\gamma_1)[1+\sqrt{1-4/(\gamma_3\gamma_2^2\gamma_1)}] > 1 \tag{42b}$$

The condition stated in Eq. (15a) in Theorem 3 is found to be the stability condition of the partial 4-th order polynomial at the both ends. When expressed in terms of CDM, as in Eq. (2c) and (24b), it can be stated as follows.

$$\gamma_2 > 1/\gamma_3 + 1/\gamma_1 \tag{42c}$$

Or

$$\gamma_2\gamma_1 > 1 + \gamma_1/\gamma_3. \tag{42d}$$

When $\gamma_1 \ge \gamma_3$, $\gamma_2\gamma_1 > 2$. Eq. (42b) is automatically satisfied. For the case $\gamma_1 < \gamma_3$, the left side of Eq. (42b) is calculated with Eq. (42d) and the relation is verified.

$$\begin{aligned} &0.5(\gamma_2\gamma_1)[1+\sqrt{1-4/(\gamma_3\gamma_2^2\gamma_1)}] \\ &= 0.5[\gamma_2\gamma_1 + \sqrt{(\gamma_2\gamma_1)^2 - 4(\gamma_1/\gamma_3)}] \\ &> 0.5[(1+\gamma_1/\gamma_3) + \sqrt{(1+\gamma_1/\gamma_3)^2 - 4(\gamma_1/\gamma_3)}] \\ &= 0.5[(1+\gamma_1/\gamma_3) + \sqrt{(1-\gamma_1/\gamma_3)^2}] = 1 \end{aligned} \quad (42e)$$

In the similar manner, $x_{Rn/2} < x_{In/2-1}$ is derived.

For $i = 2$, Eq. (38) becomes

$$-0.5(a_2/a_4)[1+\sqrt{1-4/\gamma_{22}}] < -2(a_1/a_3)[1+\sqrt{1-4/\gamma_{32}}]^{-1}. \quad (44)$$

Then from Eqs. (41b, c), $x_{R2} < x_{I1}$ is proved. By continuing the process, the following result is obtained.

$$\begin{aligned} &x_{Rn/2} < x_{In/2-1} < x_{Rn/2-1} < x_{In/2-2} \\ &< \cdots < x_{I2} < x_{R2} < x_{I1} < x_{R1} < -(a_0/a_2) < 0 \end{aligned} \quad (45)$$

Thus the roots are distinct (simple), negative, real and alternating. The root closest to 0 is x_{R1}. It was shown already that $R(0)$ and $I(0)$ are positive. Theorem 6 is satisfied and $P(s)$ is stable.

6.3 Derivation of Corollaries

Proof of Corollary 1 and 2 is briefly touched in the original paper. But main body of the proof is referred to another Russian paper. A proof in conformity with CDM will be given hereafter.

The stability condition of a 5-th order polynomial

$$P_5(s) = a_5s^5 + a_4s^4 + a_3s^3 + a_2s^2 + a_1s + a_0 \quad (46a)$$

is the same as that of the 4-th order polynomial reduced by the Routh table.

$$P_4(s) = a_4s^4 + a_3^*s^3 + a_2s^2 + a_1^*s + a_0 \quad (46b)$$

$$a_3^* = a_3 - a_2a_5/a_4 = a_3[1-(\gamma_3\gamma_4)^{-1}]$$

$$a_1^* = a_1 - a_0a_5/a_4 = a_1[1-(\gamma_1\gamma_2\gamma_3\gamma_4)^{-1}]$$

The stability condition becomes

$$\gamma_2 > \alpha/\gamma_3 + 1/(\alpha\gamma_1) \quad (46c)$$

$$\alpha = [1-(\gamma_1\gamma_2\gamma_3\gamma_4)^{-1}][1-(\gamma_3\gamma_4)^{-1}]^{-1}.$$

By rearrangement, the stability condition is given as follows. The result is the same as that given in Theorem 3 of the original paper.

$$(\gamma_4\gamma_3 - 1)(\gamma_2\gamma_1 - 1) > \gamma_4\gamma_1[1-(\gamma_1\gamma_2\gamma_3\gamma_4)^{-1}]^2 \quad (46d)$$

Suppose $\gamma_2\gamma_1 \le \gamma_4\gamma_3$, and define the following variables.

$$\lambda_1 = (\gamma_2\gamma_1)^{-1}, \quad \lambda_2 = (\gamma_3\gamma_2)^{-1} \quad (47a)$$

$$\rho = (\gamma_2\gamma_1)(\gamma_4\gamma_3)^{-1} \le 1 \quad (47b)$$

Then Eq. (46d) becomes

$$\lambda_2 \le (1-\lambda_1)(1-\rho\lambda_1)[1-\rho\lambda_1^2]^{-2}. \quad (48a)$$

If $\gamma_2\gamma_1 \ge \gamma_4\gamma_3$, use $\gamma_4\gamma_3$ instead of $\gamma_2\gamma_1$. The result of numerical analysis for variation of ρ (Fig. 5) shows that, if the next relation is satisfied, Eq. (48a) is also satisfied.

$$\lambda_2 \le \min[1-\lambda_1, (1+\lambda_1)^{-2}] \quad (48b)$$

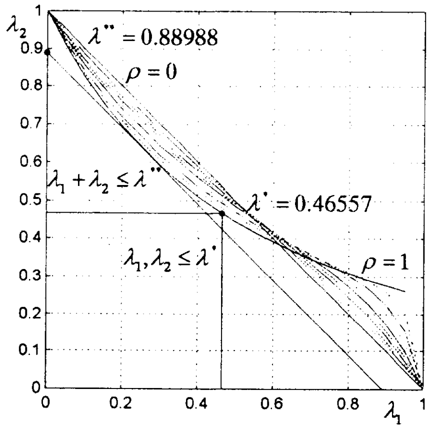


Fig. 5. Analysis of λ_2

The first term of the right hand side corresponds to $\rho = 0$, and the second term corresponds to $\rho = 1$ of Eq. (48a).

Now assume

$$\lambda_2, \lambda_1 \le \lambda^*, \quad (49a)$$

where

$$\lambda^* = (1+\lambda^*)^{-2}, \quad \text{or} \quad \lambda^* = 0.465571. \quad (49b)$$

Then Eq. (48b) and hence Eq. (48a) are satisfied. This completes the proof of Corollary 1.

Now assume

$$\lambda_2 + \lambda_1 \le \lambda^{**} \quad (50)$$

Then choose λ^{**} in such a manner that the boundary straight line of Eq. (50) is tangent to Eq. (48b). This can be obtained by solving the three equations simultaneously.

$$\lambda_2 + \lambda_1 = \lambda^{**} \quad (51a)$$

$$\lambda_2 = (1+\lambda_1)^{-2} \quad (51b)$$

$$d\lambda_2/d\lambda_1 = -2(1+\lambda_1)^{-3} = -1 \quad (51c)$$

From Eqs. (51b, c), $\lambda_2 = (0.5)^{2/3}$.

From Eq. (51c), $\lambda_1 = (0.5)^{-1/3} - 1$. Thus from Eq. (51a),

$$\begin{aligned} \lambda^{**} &= (0.5)^{2/3} + (0.5)^{-1/3} - 1 \\ &= 3/4^{1/3} - 1 = 0.889882. \end{aligned} \quad (52)$$

This completes the proof of Corollary 2.

From Fig. 5, it becomes clear that Corollary 2 which satisfies Eqs. (50) (52) is closer to the real boundary than Corollary 1, which satisfies Eqs. (49a, b). Only exception is a very small region near $\lambda_1 \approx \lambda_2 \approx \lambda^*$.

7. CONCLUSION

The major results of this paper are as follows.

(1) The sufficient conditions for stability and instability proposed by Lipatov (1978) are explained. They are modified and presented in a compact statement as Theorem 4.

(2) Brief explanation of CDM is made and the condition for stability and instability is presented in a graphical form. This graphical representation in coefficient diagram is very effective in showing the total pictures of the system to the

designer.

(3) The proofs given in the original paper are modified and expanded to improve readability.

The sufficient condition for stability and instability is least known in the control community. Only one paper (Bose, 1988) made a brief reference to it in the author's knowledge. However its practical importance seems to surpass Routh's stability criterion. It is a sincere wish of the author that the pioneering work of Lipatov is more widely accepted in the control community for the progress of control science and technology.

APPENDIX A

A1. Condition for Negative Real Roots

For a characteristic polynomial (or any polynomial with positive real coefficients) with all stability index $\gamma_i > 4$, all roots are distinct, negative, and real. This can be shown by examining the sign of P(s) for $s = -0.5(a_{i-1}/a_i)$ and $-2(a_{i-1}/a_i)$. Express $P(s)$ in the following form.

$$P(s) = a_n s^n + \cdots + a_i s^i + a_{i-1}s^{i-1} + \cdots + a_1 s + a_0$$
$$= a_{i-1}s^{i-1}[A(s)(a_i/a_{i-1})s + B(s)] \quad \text{(A1)}$$

$$A(s) = \sum_{j=0}^{n-i}(a_{i+j}/a_i)s^j \quad \text{(A2a)}$$

$$B(s) = \sum_{j=0}^{i-1}(a_{i-1-j}/a_{i-1})s^{-j} \quad \text{(A2b)}$$

Now $A(-0.5a_{i-1}/a_i)$ and $B(-0.5a_{i-1}/a_i)$ will be evaluated by replacing coefficients a_i with stability index γ_i by Eq. (3a)(6b)(6d).

$$A(-0.5a_{i-1}/a_i) = 1 + \sum_{j=1}^{n-i}(-0.5)^j/(\gamma_{i+j-1}\gamma_{i+j-2}^2\cdots\gamma_i^j)$$
$$= 1 - 0.5/\gamma_i + 0.25/(\gamma_{i+1}\gamma_i^2) - \cdots = 0.875 \sim 1 \quad \text{(A3a)}$$

$$B(-0.5a_{i-1}/a_i) = 1 + \sum_{j=1}^{i-1}(-2)^j/(\gamma_{i-1}^j\gamma_{i-2}^{j-1}\cdots\gamma_{i-j})$$
$$= 1 - 2/\gamma_{i-1} + 4/(\gamma_{i-1}^2\gamma_{i-2}) - \cdots = 0.5 \sim 1 \quad \text{(A3b)}$$

Thus for $s = -0.5(a_{i-1}/a_i)$,

$$[A(s)(a_i/a_{i-1})s + B(s)] > 0. \quad \text{(A3c)}$$

In the similar manner $A(-2a_{i-1}/a_i)$ and $B(-2a_{i-1}/a_i)$ are evaluated.

$$A(-2a_{i-1}/a_i) = 1 + \sum_{j=1}^{n-i}(-2)^j/(\gamma_{i+j-1}\gamma_{i+j-2}^2\cdots\gamma_i^j)$$
$$= 1 - 2/\gamma_i + 4/(\gamma_{i+1}\gamma_i^2) - \cdots = 0.5 \sim 1 \quad \text{(A4a)}$$

$$B(-2a_{i-1}/a_i) = 1 + \sum_{j=1}^{i-1}(-0.5)^j/(\gamma_{i-j}\gamma_{i-j+1}^2\cdots\gamma_{i-1}^j)$$
$$= 1 - 0.5/\gamma_{i-1} + 0.25/(\gamma_{i-1}^2\gamma_{i-2}) - \cdots = 0.875 \sim 1 \quad \text{(A4b)}$$

Thus for $s = -2(a_{i-1}/a_i)$,

$$[A(s)(a_i/a_{i-1})s + B(s)] < 0. \quad \text{(A4c)}$$

From Eqs. (A1)(A3c)(A4c), it becomes clear that P(s) values for $s = -0.5(a_{i-1}/a_i)$ and $-2(a_{i-1}/a_i)$ have the opposite sign. Therefore there must be at least one real root in the interval $[-2a_{i-1}/a_i, -0.5a_{i-1}/a_i]$. There are n such intervals corresponding to $i = 1 \sim n$, and such intervals do not overlap because $\gamma_i > 4$. Thus only one root is in each interval. This proves that all roots, s_i, are distinct, negative, and real.

$$-2a_{i-1}/a_i < s_i < -0.5a_{i-1}/a_i, \quad i = 1 \sim n \quad \text{(A5)}$$

A2. Interval for Negative Real Roots

The interval for the negative real roots of a characteristic polynomial $P(s)$ with $\gamma_i > 4$, as expressed in Eq. (A5), can be further narrowed as follows.

$$-\mu_i < s_i < -\eta_i, \quad i = 1 \sim n \quad \text{(A6a)}$$

$$\mu_i = 2(a_{i-1}/a_i)[1 + \sqrt{1 - 4/\gamma_i}\,]^{-1} \quad \text{(A6b)}$$

$$\eta_i = 0.5(a_{i-1}/a_i)[1 + \sqrt{1 - 4/\gamma_{i-1}}\,] \quad \text{(A6c)}$$

$$\gamma_n = \gamma_0 = \infty \quad \text{(A6d)}$$

This can be proved by the use of the relation between roots and coefficients of polynomial. Define $x_i = -s_i$, then $x_{i+1} > x_i$. Choose (n-k) of x_i s from total n x_i s, and form their products. There are $N = {}_nC_{n-k}$ such combinations, and they are named as y_{kj}. The j in y_{kj} is taken such that $y_{kj} > y_{k\,j+1}$. Clearly,

$$y_{k1} = x_n x_{n-1}\cdots x_{k+1}, \quad y_{k1} > y_{k2} > \cdots > y_{kN} \quad \text{(A7a)}$$

$$a_i/a_n = \sum_{j=1}^{N} y_{ij}, \quad N = {}_nC_{n-i} \quad \text{(A7b)}$$

$$x_i = -s_i, \quad x_n > x_{n-1} > \cdots > x_1. \quad \text{(A7c)}$$

Next, form a polynomial in $-s^2$, $PP(-s^2)$, from $P(-s)P(s)$.

$$PP(-s^2) = P(-s)P(s) = \sum_{i=1}^{n} aa_i(-s^2)^i \quad \text{(A8a)}$$

$$aa_i = a_i^2[1 + \sum_{j=1}^{M} 2(-1)^j/\gamma_{ij}], \quad M = \min(i, n-i) \quad \text{(A8b)}$$

From these relations, the upper and lower bounds of aa_i/aa_n will be sought, and some relations of y_{i1} will be obtained. The lower bound is calculated from Eqs. (A8a, b) and the condition $\gamma_i > 4$ $(i = 1 \sim n-1)$ as follows.

$$aa_i/aa_n = (a_i/a_n)^2[1 - 2/\gamma_i + 2/\gamma_{i2} - \cdots]$$
$$\geq (a_i/a_n)^2[1 - 2/\gamma_i] \quad \text{(A9a)}$$

Now the upper bound is sought. Considering the roots of $PP(-s^2)$ is $-s_i^2$, the following result is obtained using the relation of roots and coefficients of polynomial.

$$aa_i/aa_n = \sum_{j=1}^{N} y_{ij}^2, \quad N = {}_nC_{n-i} \quad \text{(A9b)}$$

This leads to

$$aa_i/aa_n = y_{i1}^2 + \sum_{j=2}^{N} y_{ij}^2 \leq y_{i1}^2 + [\sum_{j=2}^{N} y_{ij}]^2$$
$$= y_{i1}^2 + [(a_i/a_n) - y_{i1}]^2. \quad \text{(A9c)}$$

The combination of Eqs. (A9a) and (A9c) leads to the following result concerning y_{i1}.

$$y_{i1}^2 - (a_i/a_n)y_{i1} + (a_i/a_n)^2/\gamma_i \geq 0 \quad \text{(A10a)}$$

There are two solutions for Eq. (A10a).

$$y_{i1} \geq 0.5(a_i/a_n)[1 + \sqrt{1 - 4/\gamma_i}\,] \quad \text{(A10b)}$$

$$y_{i1} \le 0.5(a_i / a_n)[1-\sqrt{1-4/\gamma_i}] \quad \text{(A10c)}$$

Only upper solution Fq. (A10b) is possible, because $y_{i1} \ge 0.5(a_i / a_n)$ is proved in the following manner. From Eqs. (A7a, b) and (A9a, b), the following relation is derived and the result follows.

$$y_{i1}(a_i / a_n) = y_{i1}\sum_{j=1}^{N} y_{ij} \ge \sum_{j=1}^{N} y_{ij}^2 = aa_i / aa_n$$
$$\ge (a_i / a_n)^2[1-2/\gamma_i] \ge 0.5(a_i / a_n)^2 \quad \text{(A10d)}$$

From Eq. (A7b), it is clear that $y_{i1} \le (a_i / a_n)$. Then the relation for the interval of y_{i1} is derived as follows.

$$0.5(a_i / a_n)[1+\sqrt{1-4/\gamma_i}] \le y_{i1} \le (a_i / a_n) \quad \text{(A11a)}$$

This relation holds for $i-1$.

$$0.5(a_{i-1} / a_n)[1+\sqrt{1-4/\gamma_{i-1}}] \le y_{i-11} \le (a_{i-1} / a_n) \quad \text{(A11b)}$$

From Eq. (A7a),

$$x_i = y_{i-11} / y_{i1}. \quad \text{(A11c)}$$

Then the interval for x_i is obtained.

$$0.5(a_{i-1} / a_i)[1+\sqrt{1-4/\gamma_{i-1}}]$$
$$\le x_i \le 2(a_{i-1} / a_i)[1+\sqrt{1-4/\gamma_i}]^{-1} \quad \text{(A12)}$$

The final result Eqs. (A6a, b, c) is obtained for $s_i = -x_i$. Eq. (A6d) ($\gamma_n = \gamma_0 = \infty$) is chosen such that Eq. (A9a) holds for $i = 0$ and n, and Eq. (A12) is true for $i = 1 \sim n$.

A3. Stability Condition of the Partial 5-th Order Polynomial

A partial 5-th order polynomial of a characteristic polynomial $P(s)$ is given as $P_{5k}(s)$.

$$P_{5k}(s) = a_{k+5}s^5 + a_{k+4}s^4 + a_{k+3}s^3 + a_{k+2}s^2 + a_{k+1}s + a_k$$
$$k = 0 \sim n-5 \quad \text{(A13)}$$

This will be analyzed by Theorem 6 (Hermite-Bieler). $P_{5k}(s)$ is arranged in the even and odd terms, and s^2 is replaced by x.

$$P_{5k}(s) = R(x) + sI(x) \quad \text{(A14a)}$$

$$R(x) = a_{k+4}x^2 + a_{k+2}x + a_k \quad \text{(A14b)}$$

$$I(x) = a_{k+5}x^2 + a_{k+3}x + a_{k+1} \quad \text{(A14c)}$$

Because this polynomial is stable, the roots of $R(x), x_{R_i}$, and the roots of $I(x), x_{I_i}$, are distinct, negative, real, and alternate. They are ordered as follows.

$$x_{I2} < x_{R2} < x_{I1} < x_{R1} < 0 \quad \text{(A15)}$$

From the condition that the roots are distinct and real in the quadratic equation, γ_{k+22} and γ_{k+32} must be larger than 4. These roots are given as follows.

$$x_{R1} = -0.5(a_{k+2} / a_{k+4})[1-\sqrt{1-4/\gamma_{k+22}}] \quad \text{(A16a)}$$

$$x_{R2} = -0.5(a_{k+2} / a_{k+4})[1+\sqrt{1-4/\gamma_{k+22}}] \quad \text{(A16b)}$$

$$x_{I1} = -0.5(a_{k+3} / a_{k+5})[1-\sqrt{1-4/\gamma_{k+32}}] \quad \text{(A16c)}$$

$$x_{I2} = -0.5(a_{k+3} / a_{k+5})[1+\sqrt{1-4/\gamma_{k+32}}] \quad \text{(A16d)}$$

Eq. (16c) can be further manipulated into the following form.

$$x_{I1} = -2(a_{k+1} / a_{k+3})[1+\sqrt{1-4/\gamma_{k+32}}]^{-1} \quad \text{(A16e)}$$

From the condition that $x_{R2} < x_{I1}$, the following relation is derived.

$$-0.5(a_{k+2} / a_{k+4})[1+\sqrt{1-4/\gamma_{k+22}}]$$
$$< -2(a_{k+1} / a_{k+3})[1+\sqrt{1-4/\gamma_{k+32}}]^{-1} \quad \text{(A17)}$$

This relation holds for all $P_{5k}(s)$ ($k = 0 \sim n$-5), and the above relations are generalized as follows.

$$\gamma_{i2} > 4, \qquad i = 2 \sim n-2 \quad \text{(A18a)}$$

$$-0.5(a_i / a_{i+2})[1+\sqrt{1-4/\gamma_{i2}}]$$
$$< -2(a_{i-1} / a_{i+1})[1+\sqrt{1-4/\gamma_{i+12}}]^{-1}$$
$$i = 2 \sim n-3 \quad \text{(A18b)}$$

The above relations are the result from the stability of the partial 5-th order polynomial and will be used directly in the proof.

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