ANALYTIC WEIGHT SELECTION FOR LQ DESIGN

Shunji MANABE

Tokai University, Control Engg. Dept., 1117 Kitakaname, Hiratsuka, Kanagawa 259-1292, Japan

Abstract. The Coefficient Diagram Method (CDM) is a new design approach, whereby the designer can design the characteristic polynomial of the closed loop system efficiently taking a good balance of stability, response, and robustness. The exactly same controller can be designed by LQR, if the states are properly augmented and the weights are properly selected. These weights may not be positive definite/semi-definite and may be sign indefinite. From theses results, general rules for weights selection are derived

Key Words Control system design; control system synthesis, control theory, LQR control method, vibration.

LQ設計における解析的重み関数選定法

内容梗概 「係数図法」は制御系の新しい設計法であって、これにより閉ループ系の特性多項式を、安定性・応答性・ロバスト性の協調をとりながら、容易に設計することができる。「係数図法」と全く同じ制御器は適切な状態拡張を行い適切な重み関数（正定・半正定・不定）を用いたLQRでも求められる。この結果よりLQR・LQG最適制御における重み関数を求める一般的法則を導出した。

1. INTRODUCTION

The purpose of this paper is to show the effective method of analytically determining the weights for LQR, LQG design. From the inception of LQ development, the analytical weight selection has been the concerns of many researchers, and various efforts have been made to solve the problem (Harvey, 1978) (Stein, 1979) (Gupta 1980).

These efforts include the inverse problem (determination of the weights for given controllers) (Kalman, 1964) and the extension of weight Q beyond positive definite/semi-definite (Hayase, 1973) (Morinari, 1973) (Ohta, 1991) (Manabe, 1991). In spite of these efforts, practical methods are yet to be developed, and only workable solution at present is considered to be through iteration (Anderson, 1989, p.156).

These weights are closely related to the characteristic polynomial of the closed-loop system. If in some way the characteristic polynomial is properly determined, these weights can be derived analytically.

The proper selection of the characteristic polynomial is not difficult, if only stability and response are to be satisfied, but it becomes complicated when robustness issue is present. The coefficient diagram method (CDM) (Manabe, 1991, 1994ab, 1997abc, 1998ab) is an answer to this problem.

The strength of CDM lies in that, for any plant, minimum phase or non-minimum phase, the simplest and robust controller under practical limitation can be found. LQR and LQG sometimes fail to produce a robust controller for the plant with flexibility (poles at the vicinity of the imaginary axis) as pointed by various authors (Edmunds, 1983, and Mills, 1992). CDM produces very robust controllers in such cases. From these results, it becomes evident that these difficulty can be avoided, if the weights are properly selected and sign indefinite weights are allowed in LQ design.

In these developments, it becomes clear that LQR design with proper state augmentation is equivalent to some form of output feedback design, and it replaces LQR and LQG settings. The controller thus designed is a dynamic compensator of the lowest order. Especially when sign indefinite weights are allowed, such controller is equivalent to a CDM controller, and it includes all the controllers designed by classical control and widely used in practice.

This paper is organized as follows. In Section 2, the basics of CDM is briefly explained. In Section 3, the relations between the characteristic polynomials and the weights are examined. In Section 4, a few design examples and general rules of weights selection for simpler cases are given. In Section 5, more complicated cases are examined. It is shown that the solution of CDM is exactly equal to a LQR controller, whose states are properly augmented and the weights are properly selected. These weights may not be positive definite/semi-definite, and may be sign indefinite.

2. BASICS OF CDM

2.1 General description of CDM

The salient features of CDM will be summarized as follows;

(1) The CDM is an algebraic design method over polynomial ring in the parameter space. Instead of transfer function, its denominator and numerator are separately expressed as polynomial of s, for the plant and the controller. By so doing, the ambiguity inherent to the transfer function is avoided, and the same rigor as in the state space representation is maintained. At the same time, the compactness of expression as in the transfer function expression is retained.

(2) A special diagram called "coefficient diagram" is used as the vehicle to carry the necessary information, and as the criteria of good design. The coefficient diagram is a semi-log diagram where the coefficients of characteristic polynomial are shown in logarithmic scale in the ordinate and the numbers of power corresponding to each coefficient are shown in the abscissa, as shown in Fig. 1. The degree of convexity is a measure of stability. The general inclination of the curve is a measure of response speed. The variation of the shape of the curve is a measure of robustness. Thus the three major characteristics of control system, namely stability, response, and robustness are shown graphically in a single diagram, enabling the designer to make a balanced judgment in the course of his design.

(3) The tradition of Kessler (1960) standard form is inherited and improved. The CDM standard form is proposed. There is no overshoot in the CDM standard, while 8 % overshoot exists for the Kessler standard form.

(4) As the theoretical background of CDM, the sufficient condition of stability and instability by Lipatov (1978) is introduced. The form of expression is modified to suit to the terms of CDM. The stability can be checked graphically over the coefficient diagram.

(5) In CDM, the characteristic polynomial and the controller are designed simultaneously with due consideration to the performance specification and constraint imposed to the controller. Because of this simultaneous design nature, the designer is able to keep good balance between the rigor of the requirements and the complexity of the controller. Thus he is able to produce the simplest controller to satisfy the specification. In CDM, the performance specification is rewritten in a few parameters (stability index γ_i and equivalent time constant τ). These parameters specify the coefficients of the characteristic polynomial. These coefficients are related to the controller parameters algebraically in explicit form. These features make simultaneous design approach possible in CDM.

2.2 Mathematical relations

Some mathematical relations extensively used in CDM will be introduced hereafter. The characteristic polynomial is given in the following form.

$$P(s) = a_n s^n + \dots + a_1 s + a_0 = \sum_{i=0}^{n} a_i s^i \tag{1}$$

The stability index γ_i, the equivalent time constant τ, and stability limit γ_i^* are defined as follows.

$$\gamma_i = a_i^2 / (a_{i+1} a_{i-1}), \qquad i = 1 \sim n-1 \tag{2a}$$

$$\tau = a_1 / a_0 \tag{2b}$$

$$\gamma_i^* = 1/\gamma_{i+1} + 1/\gamma_{i-1}, \qquad \gamma_n = \gamma_0 = \infty \tag{2c}$$

From these equations the following relations are derived.

$$a_{i+1} / a_i = (a_j / a_{j-1}) / (\gamma_i \gamma_{i-1} \cdots \gamma_{j+1} \gamma_j), \qquad i \geq j \tag{3a}$$

$$a_i = a_0 \tau^i / (\gamma_{i-1} \gamma_{i-2}^2 \cdots \gamma_2^{i-2} \gamma_1^{i-1}) \tag{3b}$$

Then characteristic polynomial will be expressed by a_0, τ, and γ_i as follows.

$$P(s) = a_0 [\{\sum_{i=2}^{n} (\prod_{j=1}^{i-1} 1/\gamma_{i-j}^{j})(\tau s)^i\} + \tau s + 1] \tag{4}$$

The equivalent time constant of the i-th order τ_i and the stability index of the j-th order γ_{ij} are defined as follows.

$$\tau_i = a_{i+1} / a_i = \tau / (\gamma_i \cdots \gamma_2 \gamma_1) \tag{5}$$

$$\gamma_{ij} = a_i^2 / (a_{i+j} a_{i-j}) = [\prod_{k=1}^{j-1} (\gamma_{i+j-k} \gamma_{i-j+k})^k] \gamma_i^j \tag{6}$$

Thus τ can be considered the equivalent time constant of the 0-th order and γ_i is considered as the stability index of the 1st order. The stability index of the 2nd order is a good measure of stability and is shown below.

$$\gamma_{i2} = a_i^2 / (a_{i+2} a_{i-2}) = \gamma_{i+1} \gamma_i^2 \gamma_{i-1} \tag{7}$$

2.3 Coefficient diagram

When a characteristic polynomial is expressed as

$$P(s) = 0.25 s^5 + s^4 + 2s^3 + 2s^2 + s + 0.2\,, \tag{8}$$

then

$$a_i = [0.25 \;\; 1 \;\; 2 \;\; 2 \;\; 1 \;\; 0.2] \tag{9a}$$

$$\gamma_i = [2 \;\; 2 \;\; 2 \;\; 2.5] \tag{9b}$$

$$\tau = 5 \tag{9c}$$

$$\gamma_i^* = [0.5 \;\; 1 \;\; 0.9 \;\; 0.5]\,. \tag{9d}$$

The coefficient diagram is shown as in Fig. 1, where coefficient a_i is read by the left side scale, and stability index γ_i, equivalent time constant τ, and stability limit γ_i^* are read by the right side scale. The τ is expressed by a line connecting 1 to τ. The stability index γ_i can be graphically obtained (Fig. 2a). If the curvature of the a_i becomes larger (Fig. 2b), the system becomes more stable, corresponding to larger stability index γ_i. If the a_i curve is left-end down (Fig. 2c), the equivalent time constant τ is small and response is fast. The equivalent time constant τ specifies the response speed.

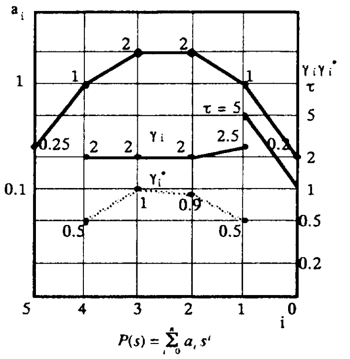


Fig. 1. Coefficient diagram

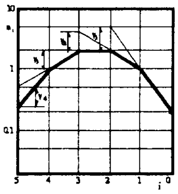


Fig.2a. γ_i

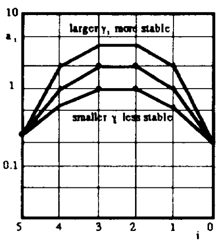
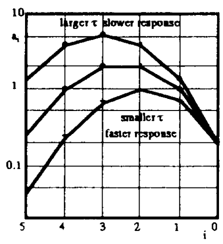


Fig. 2b. Effect of γ_i



c. Effect of τ

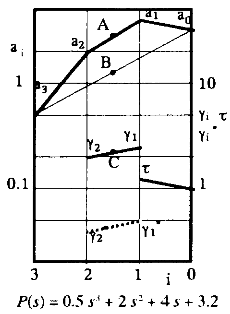
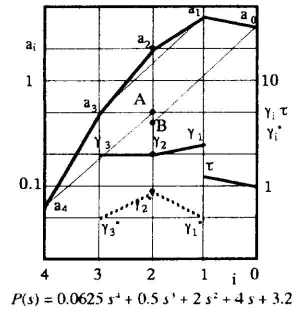


Fig. 3a. 3rd order



b. 4th order

2.4 Stability condition

From the Routh-Hurwitz stability criterion, the stability condition for the 3rd order is given as

$$a_2 a_1 > a_3 a_0 \,. \tag{10a}$$

If it is expressed by stability index,

$$\gamma_2 \gamma_1 > 1 \,. \tag{10b}$$

The stability condition for the fourth order is given as

$$a_2 > (a_1 / a_3)\, a_4 + (a_3 / a_1)\, a_0 \tag{11a}$$

$$\gamma_2 > \gamma_2^* \,. \tag{11b}$$

For the system higher than or including 5th degree, Lipatov (1978) gave the sufficient condition for stability and instability in several different forms. The conditions most suitable to CDM can be stated as follows;

"The system is stable, if all the partial 4th order polynomials are stable with the margin of 1.12. The system is unstable if some partial 3rd order polynomial is unstable."

Thus the sufficient condition for stability is given as

$$a_i > 1.12 \left[\frac{a_{i-1}}{a_{i+1}} a_{i+2} + \frac{a_{i+1}}{a_{i-1}} a_{i-2} \right] \tag{12a}$$

$$\gamma_i > 1.12\, \gamma_i^*, \quad \textit{for all } i = 2 \sim n-2 \,. \tag{12b}$$

The sufficient condition for instability is given as

$$a_{i+1} a_i \le a_{i+2} a_{i-1} \tag{13a}$$

$$\gamma_{i+1} \gamma_i \le 1, \quad \textit{for some } i = 1 \sim n-2 \tag{13b}$$

These conditions can be graphically expressed in the coefficient diagram. Fig. 3a is a 3rd-order example. Point A is $(a_2\, a_1)^{0.5}$ and point B is $(a_3\, a_0)^{0.5}$. Thus if A is above B, the system is stable. Point C is $(\gamma_2\, \gamma_1)^{0.5}$. If it is above 1, the system is stable.

Fig. 3b is a 4th-order example. Point A is obtained by drawing a line from a_4 in parallel with line $a_3\, a_1$. Similarly point B is obtained by drawing a line from a_0 in parallel with line $a_3\, a_1$. The stability condition is $a_2 > (A + B)$. The other condition is $\gamma_2 > \gamma_2^*$.

2.5 Standard form of CDM

In CDM, the recommended standard form is

$$\gamma_1 = 2.5, \quad \gamma_{n-1} = \gamma_{n-2} = \ldots = \gamma_2 = 2 \tag{14}$$

The standard form has the favorable characteristics as follows;

(1) When the order of the numerator polynomial is zero, as in type 1 servo, the system has virtually no overshoot. A proper overshoot (about 40%) is guaranteed when the numerator polynomial is selected to form a type 2 servo.

(2) Among the system with the same equivalent time constant τ, the standard form has the shortest settling time. The settling time is about $2.5 \sim 3\,\tau$.

(3) For the same equivalent time constant, the step responses of the standard form show almost equal wave forms irrespective to the order of the characteristic polynomials.

(4) The lower order poles are aligned on a vertical line. The higher order poles are located within a sector 49.5 degrees from the negative real axis, and their damping coefficient ζ is larger than 0.65.

(5) The CDM standard form is very easy to remember.

2.6 Robustness consideration

In the actual design, the choice of $\gamma_1 = 2.5$, $\gamma_2 = \gamma_3 = 2$ is strongly recommended due to stability and response requirement, but it is not necessary to make $\gamma_4 \sim \gamma_{n-1}$ equal to 2. The condition can be relaxed as

$$\gamma_i > 1.5\, \gamma_i^* . \tag{15}$$

With such freedom, designer have the freedom of designing the controller together with the characteristic polynomial, and he can integrate robustness in the the characteristic polynomial with a small sacrifice of stability and response.

It is clear from Eqs. (2c) (12b) that, if all γ_is are larger than 1.5, the system is stable. Lipatov (1978) proved, in the process of proving his main theorem, that all roots are real negative, if all γ_is are larger than 4. From these observation it is safe to say that γ_i should be chosen in a region of 1.5 ~ 4. Because the essence of the CDM lies in the proper selection of stability indices γ_i's, some experiences are required in actual design, as is true in any design effort.

3. WEIGHT POLYNOMIAL

3.1 CDM representation

The standard block diagram of the CDM design for a single-input single-output system is shown in Fig. 4 (Manabe, 1998b), where y_r, u, x, y, d, and n are reference input, input, basic state variable, output, disturbance, and noise respectively.

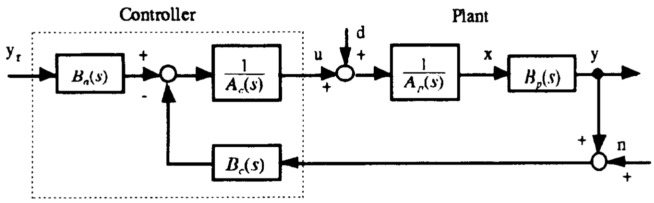


Fig. 4. CDM standard block diagram

When d and n are absent, the responses of x to u and to y_r are given as

$$A_p(s)\, x = u \tag{16a}$$

$$P(s)\, x = B_a(s)\, y_r, \tag{16b}$$

where P(s) is the characteristic polynomial and given as

$$P(s) = A_c(s)\, A_p(s) + B_c(s)\, B_p(s). \tag{16c}$$

When $B_a(s)$ is chosen as P(0), steady state value of P(s), Eq. (16a) becomes as

$$P(s)\, x = P(0)\, y_r. \tag{16d}$$

3.2 LQR formulation

The plant is expressed in the state space expression

$$\dot{x} = Ax + Bu, \tag{17a}$$

where **x** is a vector of dimension n, and **u** is a scaler. LQR design is made to minimize the performance index J given as

$$J = \int_0^\infty [\mathbf{x}^T Q\, \mathbf{x} + \mathbf{u}^T R\, \mathbf{u}]\, dt \,, \tag{17b}$$

where R is positive definite, but Q is not necessarily sign definite (Morinari,1973). The closed-loop poles of for the system with the feedback control are given by the stable eigen values of the Hamiltonian H, where no eigen values lie on the imaginary axis (Doyle, 1989).

$$H = \begin{bmatrix} A & -B R^{-1} B^T \\ -Q & -A^T \end{bmatrix} \tag{17c}$$

When the characteristic polynomial is given as in Eq. (1), the following relation is obtained.

$$P(-s)\, P(s) / a_n^2 = (-1)^n \det (s\, I_{2n} - H) \tag{17d}$$

Thus if P(s) is designed by CDM, the weight Q can be found. On the contrary, if Q is specified and LQ design is made, P(s) is obtained and it will be assessed in terms of CDM.

3.3 Squared polynomial

For a given polynomial P(s), P(-s)P(s) is a polynomial in $-s^2 = \Omega$, denoted as PP(Ω). PP(Ω) will be called the squared polynomial of P(s) hereafter, and P(s) will be called the original polynomial of PP(Ω).

$$P(-s)\, P(s) = PP(-s^2) = PP(\Omega) \tag{18}$$

If PP(Ω) has no positive real roots, there exists one original polynomial P(s) which is stable. This polynomial will be called the square-root polynomial of PP(Ω). When P(s) is a characteristic polynomial, P(s) is the square-root polynomial of the squared polynomial PP(Ω) = P(-s) P(s), because it is stable. The coefficients of these polynomials are related as follows.

$$P(s) = a_n s^n + \dots + a_1 s + a_0 = \sum_{i=0}^{n} a_i\, s^i \tag{19a}$$

$$PP(\Omega) = aq_n\, \Omega^n + \dots + aq_1\, \Omega + aq_0 = \sum_{i=0}^{n} aq_i\, \Omega^i \tag{19b}$$

$$aq_n = a_n^2\,, \qquad aq_0 = a_0^2 \tag{19c}$$

$$\begin{aligned} aq_i &= a_i^2 - 2\, a_{i+1}\, a_{i-1} + 2\, a_{i+2}\, a_{i-2} + \dots \\ &= a_i^2 + 2 \sum_{i=j}^{m} (-1)^j\, a_{i+j}\, a_{i-j} \qquad m = \min\,(i,\ n-i) \\ &= \mu\, a_i^2 \end{aligned} \tag{19d}$$

$$\mu = 1 + 2 \sum_{i=j}^{m} (-1)^j / \gamma_{i\,j} \tag{19e}$$

In this way the coefficient aq_i of PP(Ω) is expressed by the coefficient a_i and the stability index of high order $\gamma_{i\,j}$, which, in turn, is expressed by stability index γ_i as in Eq. (6). For simplicity, all γ_i's are assumed to be equal. Then

$$\gamma_{i\,j} = \gamma_i^{j \times j}, \tag{20}$$

and the ratio $\mu = aq_i / a_i^2$ for different m is calculated as in Table 1. From this table, it is clear that aq_i becomes very small for γ_i less than 2, and becomes negative in some cases.

By Eqs. (3b) (19c), the ratio aq_0 / aq_n is obtained as follows.

$$aq_0 / aq_n = (\lambda / \tau)^{2n} \tag{21a}$$

$$\lambda = (\gamma_{n-1}\, \gamma_{n-2}^2 \cdots \gamma_1^{n-1})^{(1/n)} \tag{21b}$$

For the standard form of CDM, where $\gamma_1 = 2.5$, $\gamma_2 = \gamma_3 = \dots = 2$, λ is a function of n.

$$\lambda = 2^{0.5(n-1)}\, 1.25^{(n-1)/n} \tag{21c}$$

For equal γ_i's, λ is a function of γ_i and n.

$$\lambda = \gamma_i^{0.5(n-1)} \tag{21d}$$

PP(Ω) can be normalized as $aq_n = aq_0 = 1$, when equivalent time constant τ is chosen to be λ, For the case of CDM standard form, coefficients aq_i's are calculated for various order n by computer. These values are listed in Table 2 together with λ. The actual aq_i is obtained as follows.

$$aq_i = aq_n\,(aq_0/aq_n)^{(n-i)/n}\,aq_i(normalized) \tag{22}$$

For the case of equal γ_i, aq_i(normalized) is derived from Eqs. (3b) (19c) (19d) (22) as follows.

$$aq_i(normalized) = \mu\,\gamma_i^{i(n-i)} \tag{23a}$$

Usually γ_i is chosen to be from 1.8 to 2, μ is about 0.1 as is seen from Table 1. Then

$$aq_i(normalized) \simeq 0.1\,\gamma_i^{i(n-i)}. \tag{23b}$$

In this way, the coefficients aq_i of the squared polynomial PP(Ω) are expressed in terms of the parameters used in CDM.

Table 1 The ratio $\mu = aq_i/a_i^2$

γ_i	$\mu = aq_i/a_i^2$ m=1	m=2	m=∞
4	0.5	0.50781	0.50780
2.5	0.2	0.25120	0.25068
2	0	0.12500	0.12112
1.7	-0.17647	0.062990	0.046533
1.5	-0.33333	0.061728	0.012670

Table 2 The normalized aq_i

n	aq_i (normalized)	λ
2	[1 0.5 1]	1.5811
3	[1 0 1.0772 1]	2.3208
4	[1 0 2 2.2361 1]	3.3437
5	[1 0 9.6535 8.3651 4.5731 1]	4.7818
6	[1 0 37.133 78.000 34.471 9.2832 1]	6.8129
7	[1 0 145.41 600.55 644.21 140.85 18.765 1]	9.6863
8	[1 0 572.43 4690.9 9922.0 5278.4 572.43 37.830 1]	13.753

3.4 Weight polynomial selection

The right hand side of, Eq. (17d) is further simplified in the standard manner (Kailath,1980, p.651).

$$(-1)^n \det(s\,I_{2n} - H) = \det(sI_n - A)\det(-sI_n - A^T)$$
$$[1 + B^T(-sI_n - A^T)^{-1}Q\,(sI_n - A)^{-1}B\,R^{-1}]. \tag{24}$$

In CDM, the plant is given as

$$A_p(s)\,x = u \tag{25a}$$

$$A_p(s) = ap_n\,s^n \ldots + ap_1\,s + ap_0 = \sum_{i=0}^{n} ap_i\,s^i. \tag{25b}$$

In state space expression, it becomes

$$\begin{bmatrix} \dot{x}_{n-1} \\ \dot{x}_{n-2} \\ \dot{x}_1 \\ \dot{x}_0 \end{bmatrix} = \begin{bmatrix} -ap_{n-1}/ap_n & -ap_{n-2}/ap_n & -ap_1/ap_n & -ap_0/ap_n \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_{n-1} \\ x_{n-2} \\ x_1 \\ x_0 \end{bmatrix} + \begin{bmatrix} 1/ap_n \\ 0 \\ 0 \\ 0 \end{bmatrix} u, \tag{26}$$

where x_0 is equal to the basic state variable x. Now R is chosen to 1 and Q is chosen to be a diagonal matrix.

$$R = 1 \tag{27a}$$

$$Q = \mathrm{diag}\,([q_{n-1}\ q_{n-2}\ \cdots\ q_1\ q_0]) \tag{27b}$$

Eq. (24) becomes

$$(-1)^n \det(s\,I_{2n} - H) = [AA_p(-s^2) + Q(-s^2)]\,/\,ap_n^2 \tag{28a}$$

$$AA_p(-s^2) = A_p(-s)\,A_p(s) = \sum_{i=0}^{n} apq_i\,(-s^2)^i \tag{28b}$$

$$Q(-s^2) = \sum_{i=0}^{n-1} q_i\,(-s^2)^i \tag{28c}$$

Because, for this problem, the highest coefficient of the characteristic polynomial a_n is equal to the highest coefficient of the plant denominator polynomial ap_n, Eqs. (17d) (18) (28a) give the following relation.

$$PP(\Omega) = AA_p(\Omega) + Q(\Omega) \tag{29a}$$

$$PP(\Omega) = \sum_{i=0}^{n} aq_i\,\Omega^i \tag{29b}$$

$$AA_p(\Omega) = \sum_{i=0}^{n} apq_i\,\Omega^i \tag{29c}$$

$$Q(\Omega) = \sum_{i=0}^{n-1} q_i\,\Omega^i \tag{29d}$$

When the squared polynomial of the characteristic polynomial PP(Ω) is specified, the weight polynomial Q(Ω) can be obtained by subtracting the squared polynomial of the plant denominator polynomial $AA_p(\Omega)$ from PP(Ω).

4. DESIGN EXAMPLE

4.1 Proportional control

The first example is a simple proportional control of a 2nd order system shown in Fig. 5a. The plant and controller polynomials in CDM from are given as follows.

$$A_p(s) = s\,(s+1), \quad B_p = 1 \tag{30a}$$

$$A_c(s) = 1, \quad B_c(s) = B_a(s) = k_0 \tag{30b}$$

The characteristic polynomial P(s) becomes

$$P(s) = s^2 + s + k_0 \tag{30c}$$

In CDM design, $\gamma_1 = 2.5$ and k_0 is calculated by Eq. (2a).

$$k_0 = a_1^2\,/\,(a_2\,\gamma_1) = 0.4 \tag{31}$$

For this value,

$$PP(\Omega) = \Omega^2 + 0.2\Omega + 0.16. \tag{32a}$$

Because

$$AA_p(\Omega) = \Omega^2 + \Omega, \tag{32b}$$

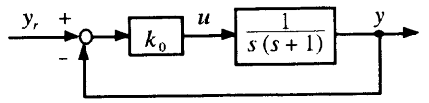


Fig. 5a. Proportional control, CDM design

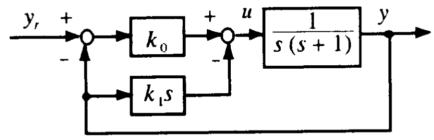
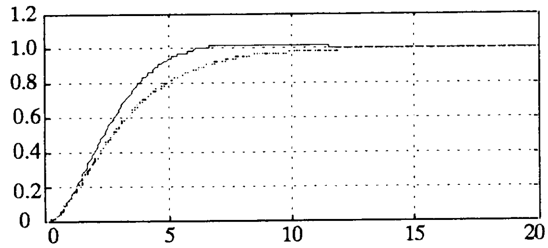


Fig. 5b. LQR design with positve semi-definite Q



— Sign indefinite Q, ---- Positeve semi-definite Q

Fig. 6. Comparison of responses

the weight polynomial becomes, by Eq. (29),

$$Q(\Omega) = -0.8\Omega + 0.16 \quad (32c)$$

In this case q_1 = -0.8 and Q is sign indefinite.

In order to avoid the sign-indefiniteness, $q_i = [q_1 \ \ q_0] = [0 \ \ 0.16]$ is selected. Then

$$PP(\Omega) = \Omega^2 + \Omega + 0.16 \quad (33a)$$

By taking the square root polynomial using a special MATLAB M-File, P(s) is obtained as

$$P(s) = s^2 + 1.3416s + 0.4 \quad (33b)$$

The controller to realize the P(s) is

$$A_c(s) = 1, \quad B_c(s) = k_1 s + k_0 = 0.3416s + 0.4,$$
$$B_a(s) = k_0 = 0.4 \quad (34)$$

The controller now becomes a PD controller and its block diagram is shown in Fig. 5b.

Fig. 6 shows the comparison of the step responses. It will be seen that the case of positive semi-definite Q is slower in response even though the control law is more complicated . This example shows the greatest shortcoming of LQR design, that is, the failure of producing even the simplest and well-accepted control law.

4.2 <u>Simplified ACC benchmark problem</u>

ACC benchmark problem (Wie, 1992) is a two-mass-spring control problem, and is used to evaluate various control design methodologies. The CDM produced a controller which is comparable to the best controller so-far reported (Manabe, 1997a). In this example, the problem is made simpler by

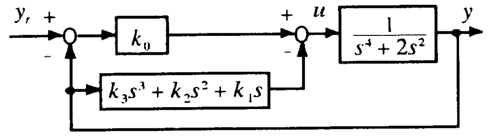


Fig. 7. Simplified ACC benchmark problem

assuming the all the states can be measured. The block diagram is shown in fig. 7. The plant and controller polynomials are given as follows.

$$A_p(s) = s^4 + 2s^2, \quad B_p(s) = 1 \quad (35a)$$
$$A_c(s) = 1, \quad B_c(s) = k_3 s^3 + k_2 s^2 + k_1 s + k_0 \quad (35b)$$
$$B_a(s) = k_0 \quad (35c)$$

The characteristic polynomial P(s) becomes

$$P(s) = s^4 + k_3 s^3 + (2 + k_2) s^2 + k_1 s + k_0. \quad (35d)$$

The selection of $k_i = [k_3 \ \ k_2 \ \ k_1 \ \ k_0]$ as below

$$k_i = [2 \ \ 0 \ \ 1 \ \ 0.2] \quad (36)$$

will produce a CDM standard controller with the narrowest bandwidth while keeping all k_i non-negative, which roughly corresponds to the non-minimum phase controller.

For this case

$$PP(\Omega) = \Omega^4 + 0.4\Omega^2 + 0.2\Omega + 0.04 \quad (37a)$$

Because

$$AA_p(\Omega) = \Omega^4 - 4\Omega^3 + 4\Omega^2, \quad (37b)$$

the weigh polynomial becomes

$$Q(\Omega) = 4\Omega^3 - 3.6\Omega^2 + 0.2\Omega + 0.04 \quad (37c)$$

Its coefficients $q_i = [q_3 \ \ q_2 \ \ q_1 \ \ q_0]$ are

$$q_i = [4 \ \ -3.6 \ \ 0.2 \ \ 0.04] \quad (37d)$$

The conspicuous point of q_i is that large q_3 term is placed in order to damp oscillation and negative q_2 is placed to prevent the system becomes overly stable. This should be the common method of selecting weights in flexible systems.

In order to prove this finding, a controller is designed for

$$q_i = [0 \ \ 0 \ \ 0.2 \ \ 0.04] \quad (38a)$$

The design results are as follows.

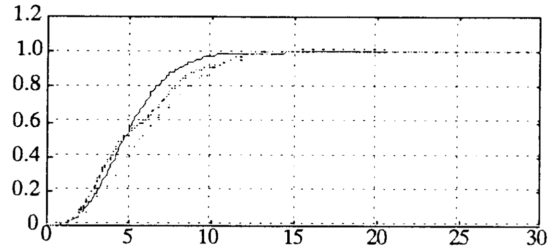
$$k_i = [0.72904 \ \ 0.26575 \ \ 1.0518 \ \ 0.2] \quad (38b)$$
$$\gamma_i = [0.23458 \ \ 6.6948 \ \ 2.4414] \quad (38c)$$

The excessively small value of γ_3 suggests the existence of a vibrational mode, -0.11390±j1.4287. Actually the vibrational zeros of the controller, -0.084955±j1.1843, are close to the plant vibrational poles, ±1.4142, with the effect of pole-zero cancellation.

In order to find the effect of negative value of q_2, a controller is designed for

$$q_i = [4 \ \ 0 \ \ 0.2 \ \ 0.04] \quad (39a)$$

The design results are as follows.



— CDM sign indefinite Q, --- Ordinary Q,

..... Positive semi-definite Q

Fig. 8. Comparison of responses

$$k_i = [2.4869 \quad 1.0923 \quad 1.1987 \quad 0.2] \tag{39b}$$

$$\gamma_i = [2 \quad 3.2076 \quad 2.3234] \tag{39c}$$

The comparatively large value of γ_2 suggests the excessive stability. Fig. 8 shows the comparison of the three cases. The CDM design gives the best response.

4.3 <u>General rules of weights selection</u>

From the results of Section 3 and the above example, general rules of specifying PP(Ω) and Q(Ω)are summarized as follows.

(1) The coefficient aq_n and aq_0 must be positive. The other aq_i should be selected to be non-negative. Then PP(Ω) is positive for any positive real Ω and PP(Ω) has no positive real roots. P(s), the square root polynomial of PP(Ω), can be always obtained. The system with this P(s) as its characteristic polynomial has the stability better than the Butterworth filter, because the Butterworth filter corresponds to the case where aq_n and aq_0 are positive and the rest of aq_i's are all zero. If a negative coefficient apq_i is found in AAp(Ω), fill it with the proper weight q_i, and make the coefficient aq_i nonnegative.

(2) Select aq_0 as follows.

$$aq_0 = aq_n\,(\lambda / \tau)^{2n}, \quad aq_n = ap_n^2 \tag{40a}$$

λ is chosen by Table 2 or Eq. (21d). τ is chosen from the requirement of the settling time t_s, where $t_s = 2.5{\sim}3\tau$. In ordinary system, $ap_0 = 0$, and

$$q_0 = aq_0. \tag{40b}$$

In some cases, the steady state gain k_0 is specified before hand. In such a case (with $ap_0 = 0$), simply select

$$q_0 = k_0^2. \tag{40c}$$

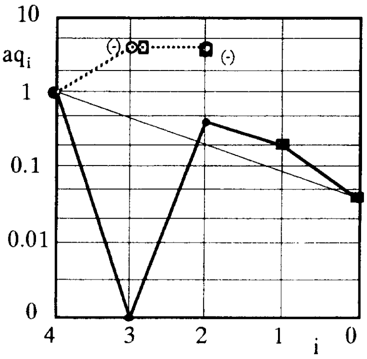
(3) Select the rest of aq_i's, such that PP(Ω) becomes a nice convex curve.

$$aq_i = aq_n\,(aq_0 / aq_n)^{(n-i)/n}\, aq_i(normalized) \tag{41a}$$

aq_i(normalized) can be obtained from Table 2 or Eq. (23b). Weights are obtained by Eq. (29a).

$$q_i = aq_i - apq_i \tag{41b}$$

The selection of aq_i is not critical, and even very approximate selection will give satisfactory results.



$AA_p(\Omega) = \Omega^4 - 4\Omega^3 + 4\Omega^2$

$n = 4,\ \lambda = 3.3437,\ \tau = 5,\ aq_n = 1,\ aq_0 = 0.04$

$PP(\Omega) = \Omega^4 + 0\Omega^3 + 0.4\Omega^2 + 0.2\Omega + 0.04$

Fig.9. Squared coefficient diagram

(4) In the PP(Ω) design a coefficient diagram of squared polynomial, which will be called as "squared coefficient diagram" hereafter, is very helpful. An example is shown in Fig. 9. First draw $AA_p(\Omega)$ and denote them by circles. When apq_i is negative simply plot the absolute value and attach (-). Plot aq_0, and connect $aq_n = apq_n$ with a straight line. Add (actually multiply) aq_i(normalized) to the point where the straight line crosses the order i vertical line. The aq_i's are plotted by dot. The difference of aq_i and apq_i is q_i and indicated by small squares.

5. LQR EQUIVALENT TO CDM

5.1 <u>CDM design</u>

An example of CDM design is given by Manabe (1998b). The block diagram is shown in FIg. 10. Plant parameters are given as

$$A_p(s) = 0.25\, s^3 + 1.25\, s^2 + s \tag{42a}$$

$$B_p(s) = 0.1\, s + 1 \tag{42b}$$

Controller is assumed to take the following form.

$$A_c(s) = l_2 s^2 + l_1 s + l_0 \tag{43a}$$

$$B_c(s) = k_2 s^2 + k_1 s + k_0 \tag{43b}$$

$$B_a(s) = k_0 \tag{43c}$$

$l_0 = 1$ and $k_0 = 20$ is given as the specification for steady state gain. Design is made such that $l_1 / l_2 = 10$, twice of the highest break point of the plant denominator, with the following results.

$$k_i = [26.488 \quad 45.496 \quad 20] \tag{44a}$$

$$l_i = [1.4750 \quad 14.750 \quad 1] \tag{44b}$$

$$a_i = [0.36876 \quad 5.5313 \quad 22.811 \quad 47.037 \quad 48.496 \quad 20.000] \tag{44c}$$

$$\gamma_i = [3.6371 \quad 2 \quad 2 \quad 2.5] \tag{44d}$$

$$\tau = 2.4248 \tag{44e}$$

Now the problem is to find the equivalent LQR problem and its weights.

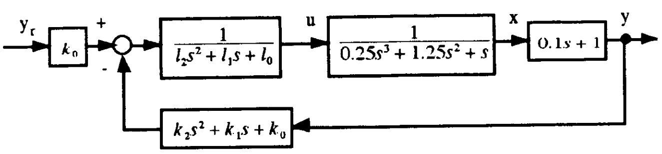


Fig. 10. Example of CDM design

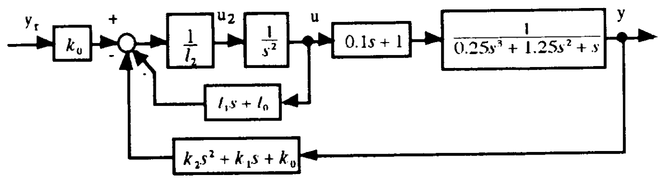


Fig. 11. State feedback representation

5.2 LQR weight eqivalent to CDM

LQR formulation and weight polynomials for the state feedback case is derived in Section 3. The CDM is for the output feedback case with a dynamic compensator. The CDM standard environment can be transformed to a state feedback case if proper state variables are chosen. An example is shown in Fig. 11. Denote the i-th derivatives of the plant input u and output y as u_i and y_i. Then u_0 = u, and y_0 = y. Now consider a state augmented system, where the highest derivative of u, u_{nc}, is considered as the plant input, and rest of u_i's and all y_i's are considered as states. The nc is the order of the controller denominator polynomial. The plant polynomial equations ares given as

$$s^{nc}\, u = u_{nc} \tag{45a}$$

$$A_p(s)\, y = B_p(s)\, u \tag{45b}$$

The augmented system is now a state feedback case. The LQR design is made to minimize the performance index J given as

$$J = \int_0^\infty [\sum_{i=0}^{nc} qu_i\, u_i^2 + \sum_{i=0}^{np-1} qy_i\, y_i^2]\, dt, \tag{46a}$$

where qu_i's and qy_i's are scaler constants, and np is the order of $A_p(s)$. In ordinary expression, the weight matrices R and Q are expressed as

$$R = qu_{nc} \tag{46b}$$

$$Q = \text{diag}\,([qu_{nc-1} \cdots qu_0\ qy_{np-1} \cdots qy_1\ qy_0]). \tag{46c}$$

The similar analysis as in Section 3 gives the result as follows.

$$PP(\Omega) = Q_u(\Omega)\, AA_p(\Omega) + Q_y(\Omega)\, BB_p(\Omega) \tag{47a}$$

$$PP(\Omega) = P(-s)P(s) = \sum_{i=0}^{nc+np} aq_i\, \Omega^i \tag{47b}$$

$$AA_p(\Omega) = A_p(-s)A_p(s) = \sum_{i=0}^{np} apq_i\, \Omega^i \tag{47c}$$

$$BB_p(\Omega) = B_p(-s)B_p(s) = \sum_{i=0}^{mp} bpq_i\, \Omega^i \tag{47d}$$

$$Q_u(\Omega) = \sum_{i=0}^{nc} qu_i\, \Omega^i \tag{47e}$$

$$Q_y(\Omega) = \sum_{i=0}^{np-1} qy_i\, \Omega^i, \tag{47f}$$

where mp is the order of the plant numerator polynomial $B_p(s)$. Thus if PP(Ω) is obtained as the result of CDM design, The weight polynomials $Q_u(\Omega)$ and $Q_y(\Omega)$ are obtained.

If the weight polynomials $Q_u(\Omega)$ and Qy(Ω) are given beforehand, there are two ways to obtain the controller. One way is by obtaining the characteristic polynomial P(s) through the square root operation of PP(Ω). The controller are obtained by solving the Diophantine equation Eq. (16c). The second way is to solve the LQR problem by R and Q in Eqs. (46b) (46c). In either cases the results are the same. However the first way is much more convenient, because a special MATLAB M-file has been developed for this purpose.

In actual practice, PP(Ω) design by a squared coefficient diagram is more convenient than specifying $Q_u(\Omega)$ and Qy(Ω). The controller design based on the design of PP(Ω) is called as "squared coefficient diagram method" (s-CDM). As the design approach, the ordinary CDM is recommended, but s-CDM can be used as a complementary approach, because of its close relation with LQ design.

5.3 Design verification

In order to verify these results, numerical computations are made. First CDM design is made by using a special MATLAB M-file called "gkc". The command sequence is given below.

```
ap=[0.25 1.25 1 0]; bp=[0.1 1]; gr=[2 2 2 2.5];
nc=2; mc=2; t=2.4248; k0=20; tm=0.5; gkc        (48a)
```

The design results are the same as Eqs. (44a, b, c, d, e).

Using a_i, denoted as aa inMATLAB, aq_i is obtained by making its squared polynomial using "a2aq". Then weight polynomial and controller parameters are obtained by "aqwc".

```
aq=a2aq(aa); aqwc                                (48b)
```

The exactly the same controller is obtained.

The coefficients of the squared polynomial and weight polynomials are as follows.

aq_i = [0.13598 13.771 35.766 221.25 470.37 400.00] (49a)

apq_i = [0.0625 1.0625 1 0] (49b)

bpq_i = [0.01 1] (49c)

qu_i = [2.1757 183.35 -3108.3] (49d)

qy_i = [3304.7 3574.7 400.00] (49e)

It is worthy to note that there is a large negative weight qu_0 for u^2 term. Such a weight violates the positive definiteness and also is very difficult for for the designer to think of at the design phase.

Using the weight polynomial, the standard LQR design is made by MATLAB.

```
R=qu(1),
```

```
Q=diag([qu(2:size(qu,2)) qy]),
a=[0 0 0 0 0;  1 0 0 0 0;  [0.1 1 -1.25 -1 0]/0.25;
       0 0 1 0 0;  0 0 0 1 0],
b=[1;0;0;0;0],
[K,S,E]=lqr(a,b,Q,R),  kk=[1 K]/K(2)                    (50)
```

The results are as follows.

$$K = [9.9998 \ \ 0.67796 \ \ 17.958 \ \ 30.844 \ \ 13.559] \quad (51a)$$

$$kk = [1.4750 \ \ 14.750 \ \ 1 \ \ 26.488 \ \ 45.496 \ \ 20.000] \quad (51b)$$

Eq. (51a) corresponds to the following controller equation.

$$u_2 = -[9.9998u_1 + 0.67796u_0 + 17.958y_2 + 30.844y_1 + 13.559y_0] \quad (52a)$$

This will be normalized to make the coefficient for u_0 equal to one. The result is Eq. (51b), and it corresponds to the following controller equation.

$$(1.4750\, s^2 + 14.750\, s + 1)\, u = -(26.488\, s^2 + 45.496\, s + 20.000)\, y \quad (52b)$$

The results are the same as Eqs. (44a, b).

6. CONCLUSION

In this paper, an analytical weight selection method for LQ, based on CDM, is proposed, and various design examples are given. The important results are summarized as follows.

(1) The result of the CDM design gives PP(Ω), whose coefficients aq_i's are all positive and the squared coefficient diagram is of a nice convex form.

(2) Weights for LQR design can be deduced from PP(Ω), but the values are usually sign indefinite. It is almost impossible to find weights first, because they are so inconsistent.

(3) PP(Ω) can be designed directly over the squared coefficient diagram. If PP(Ω) is designed the controller is obtained immediately. Such design method will be called s-CDM (squared coefficient diagram method). This method capitalizes on the fact that the selection of aq_i is less sensitive than that of a_i, and may be used in complementary manner to CDM.

(4) The current LQ difficulty in the flexibility control is found to be due to improper weight selection, which can be easily remedied.

(5) The result of CDM is equivalent to a LQR controller with proper state augmentation and with proper weights. Such weights are usually sign indefinite.

In the course of this development, it becomes clear that H_∞ control design is interpreted simply as the different method of defining PP(Ω). This problem will be left for the future.

REFERENCES

Anderson, B. D. O. (1967). An algebraic solution to the spectral factorization problem. *IEEE Trans. Automat. Contr.*, **AC-12**, **4**, pp. 410-414.

Anderson, B. D. O. and J. B. Moore (1989). *Optimal control.* Prentice-Hall, NJ.

Bose, N. K., Jury, E. I., and Zeheb, E. (1988). On robust Hurwitz and Schur polynomials. *IEEE Trans. Automat. Contr.,* **33**, **12**, 1166-1168.

Chen, C. T. (1987). Introduction to the linear algebraic method for control system design. *IEEE Contr. Syst. Mag.*, **7**, **5**, 36-42

Chestnut, H. and R. W. Mayer (1951). *Servomechanism and regulating system design.* Vol. 1, Chap. 14, John Wiley

Doyle, J. C., K. Glover, P. P. Khargonekar, and B. A. Francis.(1989). State-space solution to standard H_2 and H_∞ control problems. *IEEE Trans. Automat. Contr.,* **34**, **8**, 831-847.

Edmunds, R. S. and D. L. Mingori (1983). Robust control system design techniques for large flexible space structures having non-collected sensors and actuators. AIAA Paper 83-2294, AIAA Guidance and Control Conference, Aug.15-17, 1983

Franklin, G. F., J. D. Powell, and Abbas Emami-Naeini (1994). *Feedback Control of Dynamic Systems.* Addison-Wesley.

Graham, D. and R. C. Lathrop (1953). The synthesis of "optimum" transient response: criteria and standard forms. *AIAA Transactions*, **72**, pt. II, 273-288

Gupta, N. K. (1980). Frequency-shaped cost functionals: extension of linear-quadratic-gaussian design methods. *AIAA Journal of Guidance, and Control,* **3**, **6**, 529-535

Harvey, C. A. and G. Stein (1978). Quadratic weights for asymptotic regulator properties. *IEEE Trans. Automat. Contr.,* **AC-23**, **3**, 378-387.

Hayase, M. (1973). Linear constant coefficient optimum control problems in which the weighting matrices are not limited to be positive semi-definite. *Trans. of SICE*, **9**, **3**, 311-319.

Hori, Y. (1994). 2-mass system control based on resonance ratio control and Manabe polynomials. Proc. of First Asian Control Conference, Tokyo, July 27-30, 1994, Vol. 3, III-741-744.

Kailath, T. (1980). *Linear Systems*, pp. 306-310, Prentice-Hall.

Kalman, R. E. (1964). When is a linear control system optimal?. *Trans. ASME*, **D86**, 51-60

Kessler, C. (1960). Ein Beitrag zur Theorie mehrschleifiger Regelungen. *Regelungstechnik*, **8**, **8**, 261-166.

Kitamori, T. (1979). A method of control system design based upon partial knowledge about controlled processes. *Transaction of SICE*, **15**, **4**, 549-555.

Lipatov, A. V. and N. I. Sokolov (1978). Some sufficient conditions for stability and instability of continuous linear stationary systems. translated from *Automatika i Telemekhanika*, No. 9, pp. 30- 37, 1978; in *Automat. Remote Contr.*, **39**, 1285-1291, 1979.

Manabe, S. (1991). Unified interpretation of classical, optimal, and H∞ control. *Journal of SICE*, **30**, **10**, 941-946.

Manabe, S. (1994a). A low-cost inverted pendulum system for control system education. the 3rd IFAC symposium on advances in control education, Tokyo, August 1-2, 1994

Manabe, S. (1994b). Coefficient diagram method as applied to the attitude control of controlled-bias-momentum satellite. 13th IFAC Symposium on Automatic Control in Aerospace, Sept. 12-16, 1994, Palo Alto, CA, 322-327.

Manabe, S. (1997a). The application of coefficient diagram method to the ACC benchmark problem. 2nd Asian Control Conference, July 22-25, 1997, Seoul, II-135-138.

Manabe, S. (1997b). The sufficient condition for stability by Lipatov and its application to the coefficient diagram method. SICE'97, July 29-31, Tokushima, I-115-116.

Manabe, S. (1997c). A simple proof and the physical interpretation of Routh stability criterion. *J. IEE Japan*, **117**, **12**, 851-854.

Manabe, S. (1998a). Controller design of two-mass resonant system by coefficient diagram method. *T. IEE Japan*, **118-D**,**1**, 58-66.

Manabe, S. (1998b). Coefficient diagram method 14th IFAC Symposium on Automatic Control in Aerospace, Aug. 24-28, 1998, Seoul, to be published.

Mills, R. A. and A. E. Bryson (1992). Parameter-robust control design using a minimax method. *AIAA Journal of Guidance, Control, and Dynamics*, **15**, **5**, 1068-1075

Morinari, B. P. (1973). The stable regulator problem and its inverse. *IEEE Trans. Automat. Contr.*, **AC-18**, **5**, pp. 454-459.

Ohta, H., M. Kakinuma, and P. N. Nikiforuk (1991). Use of negative weights in linear quadratic regulator synthesis. *AIAA Journal of Guidance, Control, and Dynamics*, **14**, **4**, 791-796

Stein, G. (1979). Generalized quadratic weights for asymptotic regulator properties. *IEEE Trans. Automat. Contr.*, **AC-24**, **4**, 559-566.

Tustin, A. and others (1958). Position control of massive objects. *Proc. IEE*, **105**, *Part C, Supplement No. 1*, 1-57

Wie, B and D.S.Bernstein (1992). Benchmark Problems for Robust Control Design. *AIAA Journal of Guidance,Control, and Dynamics*, **15**, **5**, 1057-1059